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DISCRETIZATION OF DIFFERENTIAL GEOMETRY FOR COMPUTATIONAL
GAUGE THEORY

BY

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DISSERTATION

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Abstract

This thesis develops a framework for discretizing field theories that is independent of the chosen coordinates of the underlying geometry. This independence enables the framework to be more easily utilized in a variety of domains such as those with non-trivial geometry and topology. To do this, we build on discretizations of exterior calculus including Discrete Exterior Calculus and Finite Element Exterior Calculus. We apply these methods to discrete differential geometric objects by providing a new definition of the discrete exterior derivative on dual cochains, allowing us to incorporate more general boundary conditions and prove a discrete version of adjointness of the discrete exterior derivative and the codifferential. We also provide a definition of fundamental constructions of discrete vector bundles such as the Whitney sum, tensor bundle and pullback bundles, and a definition of a discrete covariant exterior derivative on general vector-valued k -cochains that extends to endomorphism-valued cochains while leading to discrete analogs of properties of endomorphism-valued forms. As part of our investigations of discrete vector bundles, we consider the problem of under what conditions the structure group of a discrete vector bundle can be simplified and give algorithms to perform the reduction when such a reduction is possible.

We also develop discrete variational mechanics deriving the Euler-Lagrange equations for both fully-discrete (both space and time are discretized) as well as semi-discrete (space is discretized and time is left smooth) theories with and without gauge symmetries. We further derive a discrete analog of Noether's theorem and define discrete analogs of conserved current and charge densities. We apply our discretization scheme to classic examples including complex scalar field theory and electrodynamics as well as to non-Abelian Yang-Mills. Our last application is to Abelian Chern-Simons, where we consider fully- and semi-discrete discretizations utilizing both primal and dual complexes to provide simpler discrete descriptions of physical quantities and demonstrate our ability to recover other topological properties of smooth theories. In examining discretizations of topological charge, we extend a definition of the first Chern class to all vector bundles, and in addition we discuss possible discretization of the second Chern class. Finally, we consider a generalization of the Cheeger-Buser inequalities to a "hockey puck shaped" domain in \mathbb{R}^3 , showing how the eigenvalues of the one-form Laplacian change as the hockey puck shape approaches that of a solid torus. Our framework for discretizing field theories enables broader use of techniques in exterior calculus to improve numerical methods for solving physical and geometric systems.

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Chapter 1.

Introduction

Differential geometry is a valuable language for describing physical systems ranging from classical mechanics to modern field theories. A principal advantage is the ability to express quantities and operators in a coordinate-independent language, which eliminates the need for carefully constructed coordinate systems. In the discrete context this means that a system of discrete differential geometry can be carried between different domains without the need to derive a new discretization for every domain. The most fundamental objects are those of exterior calculus: the exterior derivative, hodge star, wedge product and differential form which generalize vector calculus to more general manifolds. There are two primary discretizations of exterior calculus: discrete exterior calculus (DEC) [Hirani 2003] and finite element exterior calculus (FEEC) [Arnold, Falk, and Winther 2006; Arnold, Falk, and Winther 2010] which differ in their definition and interpretation of the discrete spaces. We will be primarily building on DEC, though large parts of our work are equally applicable to either theory.

Discretizing modern physics requires the discretization of vector bundles and the objects such as vector-valued differential forms, covariant derivatives (or connections) and curvature. Modern physics can be almost entirely phrased in terms of vector bundles. In classical mechanics the most important vector bundles are the tangent and cotangent bundles which are where velocities or momenta, respectively, of the system reside. General relativity placed even greater importance on the (co)tangent bundle as the fundamental question is to determine the metric, information which can be found by examining the curvature of these bundles.

In quantum mechanics a variety of different vector bundles are common. For example, the Schrödinger equation is a partial differential equation whose solution gives a function from a base space to a one-dimensional complex vector bundle. In the path integral formulation of quantum field theory, all of the matter particles can be described as functions from the base space to an appropriate vector bundle, while the force carriers are manifestations of curvature of these bundles.

However most physical systems of interest are not solvable analytically. For instance, quantum chromodynamics (QCD) is not perturbative in the low-energy regime and therefore much of the advancement in understanding low-energy QCD has come from lattice methods [Creutz 1986]. In fact, Wilson introduced lattice gauge theory to demonstrate that quarks would be confined in QCD [Wilson 1974]. Strongly coupled systems are also common in condensed matter and thus

lattice methods have been used to try to better understand these systems [Ichinose and Matsui 2014].

Much of the literature in discrete methods has focused on lattice methods. However simplicial methods, which use generalizations of triangles instead of cubes as their building blocks, have also been developed [Ardill et al. 1983; Bender, Milton, and Sharp 1985; Bender and Milton 1986; Cahill and Reeder 1986; J. Drouffe and K. Moriarty 1983; J. M. Drouffe and K. J. M. Moriarty 1984; J. M. Drouffe, K. J. M. Moriarty, and C. Mouhas 1983; J. M. Drouffe, K. J. M. Moriarty, and C. N. Mouhas 1984]. Although the geometry can be more complicated in simplicial meshes, simplicial numerical methods have the advantage of allowing for local refinement of meshes which saves computational resources and allows for better descriptions of interesting area such as gluon flux tubes between quarks, or in regions near impurities in condensed matter systems. Also, simplicial methods are better adapted to solving problems on difficult geometries and especially spaces with non-trivial topology. In addition, simplicial meshes have more face orientations which means breaking rotational invariance is less of an issue [Celmaster 1982] and also should help with analyzing the spin structure [J. Drouffe and K. Moriarty 1983].

However, these earlier methods were derived from an axillary hypercube lattice [J. Drouffe and K. Moriarty 1983] or from a tiling [Cahill and Reeder 1986]. Recently, Christiansen et al. used finite element spaces from Finite Element Exterior Calculus to develop a simplicial gauge theory (SGT) in order to apply SGT to a wider range of possible meshes [Christiansen and Halvorsen 2011; Christiansen and Halvorsen 2012]. Some recent progress includes proving consistency and Noether's theorem for a discrete Yang-Mills action defined on a simplicial spatial domain [Christiansen and Halvorsen 2012], demonstrating that Lie-algebra Whitney forms are not gauge invariant [Christiansen and Winther 2006], and proving consistency of a Yang-Mills action on a space-time domain [Halvorsen and Sørensen 2013].

We build on the discretization of differential geometry to model physics by providing a new definition of the dual exterior derivative which allows us to better incorporate more general boundary conditions and provide a proof of adjointness of the discrete exterior derivative and codifferential. We use this tool to derive the Euler-Lagrange equations for both semi-discrete, those theories where space is discretized and time is left continuous, and fully-discrete, where both space and time are discretized; as well as a discrete analog of Noether's theorem.

Our work on discretizations of vector bundles provides a definition of the discrete covariant derivative which generalizes the usual definition of covariant derivative in the literature to general vector-valued cochains. With this tool we are able to produce discrete versions of the smooth properties of these objects. This mathematical framework allows us to consider gauge theories and derive Euler-Lagrange equations for pure gauge theories as well as those coupled to bosonic fields. We specifically investigate Yang-Mills theory, obtaining the equations of motion as well as soliton equations. We also consider the case of a charged bosonic field coupled to a $SU(n)$ gauge field and derive the conserved currents. We further investigate when the structure group of a discrete vector bundle can be simplified and give algorithms to perform the reduction when such a

reduction is possible. Specifically, we consider the problem of trivializing a discrete bundle, finding the maximum trivial subbundle of a given discrete bundle and when a bundle can be decomposed into a direct sum of bundles.

We use our tools to examine two discretizations of AbelianBF theory: the semi-discrete theory is inspired by the Chern-Simons discretization [Sun, Kumar, and Fradkin 2015] and the fully-discrete theory originally investigated by [Sen et al. 2000]. We are able to show that our discretizations are gauge-invariant and that we can recover the classical equations of motion as well as topological properties. Topological charges also play an interesting role in novel condensed matter systems and we give a definition of the first Chern class of a vector bundle as well as show that it has discrete analogs of the properties from the smooth setting. Our final Chapter reports on our work to generalize the Cheeger and Buser inequalities to a the one-form Laplacian. We are able to derive an upper bound on the smallest non-trivial eigenvalue of the one-form Laplacian on this domain which shows that the lowest eigenvalue becomes smaller as the hockey puck is “squeezed” to approach a solid torus, showing the eigenvalues “anticipating” the arising topology.

Chapter 2.

Preliminaries

In this thesis we will be developing coordinate-independent discretizations of fundamental objects in field theories and especially gauge theory. In the smooth setting these theories build on differential geometry and exterior calculus and definitions and details on theorems are available in a variety of standard references [Abraham, Marsden, and Ratiu 1988; Kobayashi and Nomizu 1996]. Our work builds on two different discretizations of exterior calculus, namely *discrete exterior calculus* (DEC) [Hirani 2003] and *finite element exterior calculus* (FEEC) [Arnold, Falk, and Winther 2006; Arnold, Falk, and Winther 2010]. While there are important differences to these two discretization schemes, the practical distinction is in the definition and interpretation of the Hodge star. These come from the inner product on the space of discrete forms and after presenting the FEEC and DEC Hodge stars we also discuss the topological Hodge star which is useful in discretizations of topological theories such as Chern-Simons or BF. Also important to our later discussion is the discretization of the exterior derivative and its codifferential since that allows for the discretization of the Laplacian.

We then review discretizations of gauge theory. After defining a discrete vector bundle we discuss known operators of importance such as such as the discrete covariant derivative, curvature and holonomy. We then discuss the space of observables, most important among them in the Wilson loop. Lastly, we introduce Yang-Mills and BF theories introducing two discretizations of the latter.

2.1. Cellular and Simplicial Complexes

For much of this thesis we will not need to distinguish between cellular (or CW) and simplicial complexes, however for notational simplicity some proofs are given in terms of simplices. The main advantage of cellular complexes is flexibility. Since the cells can be of any shape, they can be used to model systems that have a known arbitrary discrete structure. Simplices are more rigid, which is useful when defining some discrete geometric operators such as the hodge star.

Definition 2.1.1. A **CW complex** is a Hausdorff space X together with a partition of X into open cells (of perhaps varying dimension) which also satisfy:

1. For each n -dimensional open cell σ in the partition of X , there exists a continuous map f

from the n -dimensional closed ball to X such that the restriction of f to the interior of the closed ball is a homeomorphism onto the cell σ , and

2. the image of the boundary of the closed ball is contained in the union of a finite number of elements of the partition, each having cell dimension less than n .

A CW complex is called *regular* if for each n -dimensional open cell σ in the partition of X , the continuous map f from the n -dimensional closed ball to X is a homeomorphism onto the closure of the cell σ . We will only be interested in regular, piecewise linear CW complexes (that is those whose boundaries are made of straight lines), and will refer to these simply as cellular complexes. We will also denote a cell by listing the vertices that make it, i.e. $\sigma = [01234]$. A k -simplex is a special type of k -cell that economizes the number of vertices it has:

Definition 2.1.2. A k -simplex is the convex hull of $k + 1$ affinely independent points.

By affinely independent we mean that in any coordinate chart, $\phi : U \rightarrow \mathbb{R}^n$, the vectors $\{\phi(v_i) - \phi(v_0)\}$ are linearly independent. Note that while a k -cell can have an arbitrary number of vertices a k -simplex can only have exactly $k + 1$ simplices. Also note that the choice of ordering of the vertices induces an orientation on the simplex (which is specified only up to even permutations).

Definition 2.1.3. A **simplicial complex** K is a collection of simplices such that:

1. If σ^{k-1} is a face of $\sigma^k \in K$, then $\sigma^{k+1} \in K$ and
2. If $\sigma_1^k, \sigma_2^k \in K$, then $\sigma_1^k \cap \sigma_2^k$ is either a face of both simplices or the empty set.

We will also need to be able to map between cellular complexes, for instance we may wish to transform the base cellular complex to some new one, or transfer the fiber bundle over a given simplicial complex to a fiber bundle over a different cellular complex.

Definition 2.1.4. Given two cellular complexes K and L a map $\phi : K \rightarrow L$ is called a **cellular map** if $\phi(K_0) \subset L_0$ and whenever v_0, v_1, \dots, v_k are vertices of K that span a cell then $\phi(v_0), \phi(v_1), \dots, \phi(v_k)$ are vertices that span a cell of L .

2.1.1. Primal and Dual Complexes and Orientation

To any cell-complex we can assign a **dual complex** by associating a new vertex to every top dimensional cell, connect these with an edge that passes through the codimension-one cells, and so on. Given a cell σ , we will denote its dual $\star\sigma$. Also note that the dual complex to a simplicial complex is rarely a simplicial complex. Two popular methods are barycentric and circumcentric duality, where the dual vertex is placed in the barycenter or circumcenter of the top dimensional cell, respectively.

In our work all of our complexes will be **oriented** that is given a consistent orientation. This requires additional information to be given about the top-dimensional cells as well as the boundary. A typical choice in three-dimensions is that the boundary be oriented “outwards.” From the orientation of the primal complex the dual complex can be oriented as described in [Hirani 2003]

2.1.2. Cochains and Differential Forms

In order to discretize exterior calculus we need to discretize differential forms. Recall that a differential k -form is an antisymmetric $\binom{0}{k}$ -tensor and that these can be integrated on k -dimensional subsets of the ambient manifold M .

In a simplicial complex, there is a filtration by simplex dimension that provides k -dimensional subsets on which we can integrate our differential forms. We call set of k -simplices along with their boundaries the k -skeleton of the simplicial complex and denote it K_k . We will denote the number of simplicies of dimension k as N_k so $|K_k| = \sum_{i=0}^k N_i$

This motivates using k -cochains, linear maps from the k -skeleton to \mathbb{R} (or \mathbb{C}) as the discrete analog of differential k -forms. We denote the space of k -cochains as $C^k(K, \mathbb{R})$ (or $C^k(K, \mathbb{C})$ for the complex case), though we may exclude the target space if it is clear from context. Likewise, we will denote the space of dual cochains as $D^k(\star K, \mathbb{R})$ ($D^k(\star K, \mathbb{C})$ for the complex case); we will also denote dual cochains with a superscript astrisk, i.e. $\beta^* \in D^k(\star K, \mathbb{R})$.

On every k -simplex there is a boundary map ∂ which is defined as:

$$\partial \sigma^k = \partial [v_0, v_1, \dots, v_k] = \sum_i (-1)^i [v_0, v_1, \dots, \hat{v}_i, \dots, v_k]$$

where \hat{v}_i means exclude vertex v_i . Recall that for the boundary map $\partial \partial = 0$. We then define the discrete exterior derivative on a k -cochain as:

$$\langle d^k \alpha, [v_0, v_1, \dots, v_{k+1}] \rangle = \langle \alpha, \partial [v_0, v_1, \dots, v_{k+1}] \rangle$$

The discrete exterior derivative can be thought of as a matrix of size $N_{k+1} \times N_k$, which we will also denote by d^k . The discrete exterior derivative can be defined similarly for the dual complex, where the dual boundary operator is defined by [Hirani 2003]:

Definition 2.1.5. The **dual boundary operator** is defined as:

$$\partial \star [v_0, v_1, \dots, v_k] = \sum_{\sigma^{n-p-1} \succ \sigma^{n-p}} \star s_{\sigma^{n-p-1}} \sigma^{n-p-1}$$

For $0 \leq p < n$, $s_{\sigma^{n-p-1}}$ is chosen so that the induced orientation of $s_{\sigma^{n-p-1}} \sigma^{n-p-1}$ on σ^{n-p} matches that of σ^{n-p} . When $p = n$ s_{σ^1} is chosen so that the orientation of $\star(s_{\sigma^1} \sigma^1)$ is the same as the orientation that is induced on the Voronoi dual of σ^1 by the Voronoi dual of σ^0 .

As in the case of the primal discrete exterior derivative, we define the dual discrete exterior derivative for a dual $(n-k)$ -cochain β^* as:

$$\langle d_{n-k}^{\text{dual}} \beta^*, \star \sigma^k \rangle := \langle \beta^*, \partial \star \sigma^k \rangle$$

Which, thought of as a matrix operator, the discrete dual exterior derivative can be written as $d_{n-k}^{\text{dual}} = (-1)^{n-k} (d^{n-k})^T$.

The dual boundary operator does not include the entire boundary of dual cells that intersect the boundary. For example in the part of the complex shown in Figure 2.1, there are no dual edges to “close the loop” of this cell and so the dual boundary operator does not give the full boundary. We address this in Chapter 3 by including additional boundary dual cells on the boundary.

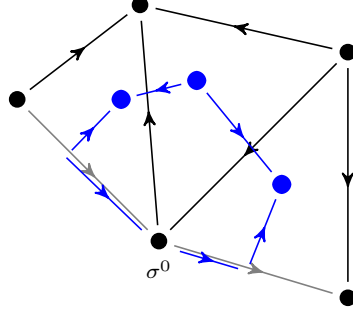


Figure 2.1.: Boundary to a dual two-cell in two dimensions.

2.1.3. Wedge Product

Multiplication of differential forms is achieved through the *wedge product*. The smooth wedge product has three key properties, namely that for any three differential forms α , β , and γ :

1. $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$ (Graded anti-commutativity)
2. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (Associativity)
3. $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|\beta|} \alpha \wedge (d\beta)$ (Leibniz rule).

Unfortunately, there is a “no go” folk theorem for discrete wedge products which states that all three of these properties cannot be simultaneously held. Our wedge products will then opt to lose associativity, though in for a restricted types of cochains we may still have associativity (see [Hirani 2003]).

We will be interested in two different types of wedge products: a primal-primal wedge product defined on simplicial complexes and a primal-dual wedge product that is defined on cell complexes.

Our primal-primal wedge product is from [Hirani 2003].

Definition 2.1.6. Given $\alpha \in C^k(K)$ and $\beta \in C^l(K)$ the **discrete primal-primal wedge product** is defined as:

$$\langle \alpha \wedge \beta, \sigma^{k+l} \rangle = \frac{1}{(k+l+1)!} \sum_{\tau \in S_{k+l+1}} (-1)^{|\tau|} \langle \alpha \smile \beta, \tau(\sigma) \rangle ,$$

where \smile is the cup product.

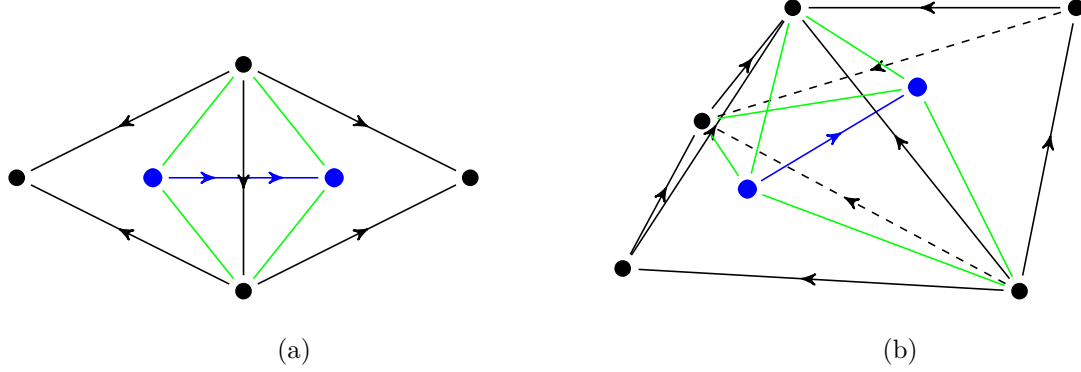


Figure 2.2.: (a) V_{σ^1} in a two-dimensional complex. The black edges are primal edges and the blue edges are dual edges. The green lines denote the boundary of the diamond region. (b) V_{σ^2} in a three-dimensional complex. The black edges are primal edges and the blue edges are dual edges. The green lines denote the boundary of the diamond region.

An important special case is when α is a zero-cochain and β is a one-cochain, and then primal-primal wedge product reduces to:

$$\langle \alpha \wedge \beta, [v_i, v_j] \rangle = \left(\frac{\langle \alpha, [v_i] \rangle + \langle \alpha, [v_j] \rangle}{2} \right) \langle \beta, [v_i, v_j] \rangle .$$

There is also a primal-dual wedge product defined between a primal k -cochain and dual $(n - k)$ -cochain [Desbrun et al. 2003]. This product takes values on the “diamond shaped regions” that are made of the intrinsic convex hull of a primal k -cell and dual $(n - k)$ -cell and are denoted as V_{σ^k} or $V_{\star\sigma^k}$, see Figures 2.2a and 2.2b. We call these regions the diamond regions because of the diamond shape of the region for the primal-dual wedge product between a primal one-cochain and dual one-cochain in two-dimensions as shown in Figure 2.2a.

Definition 2.1.7. Given a primal k -cochain α and dual $(n - k)$ cochain β^* the **top-dimensional primal-dual wedge product** is defined as:

$$\langle \alpha \wedge \beta^*, V_{\sigma^k} \rangle := \frac{1}{n} \langle \alpha, \sigma^k \rangle \langle \beta^*, \star\sigma^k \rangle$$

This definition, however fails to account for the orientation of the convex hull of σ^k and $\star\sigma^k$ and because of that is not graded-anti-commutative. In Section 3.3, we define the orientation of V_{σ^k} and $V_{\star\sigma^k}$ and therefore enhance this definition by making it graded-anti-commutative. Furthermore we define a **co-dimension one primal-dual wedge product** and examine how these primal-dual wedge products interact with the discrete exterior derivative.

2.1.4. Hodge Stars and Discrete Inner Product

In the smooth setting the Hodge star comes from the inner product on differential forms by the formula:

$$\alpha \wedge \star \beta = \langle \alpha, \beta \rangle \mu ,$$

where α and β are k -forms, \langle , \rangle is the inner product on forms and μ is the volume form. In the discrete setting the Hodge stars play an analogous role, defining the integrated inner product between cochains:

Definition 2.1.8. Given a primal k -cochain α and primal k -cochain β the **primal inner product** is given by:

$$(\alpha, \beta) := \alpha^T \star_k \beta .$$

Likewise, the **dual inner product** is given by:

$$(\alpha^*, \beta^*) := (\alpha^*)^T \star_k^{-1} \beta^* .$$

for any dual $(n - k)$ -cochains α^* and β^* .

Throughout we will use rounded parenthesis, $(,)$ for integrated inner products and pointed \langle , \rangle for local evaluation, such as the value a cochains when evaluated on a particular cell.

We have already stated that the difference between discrete Hodge stars is the primary practical difference between FEEC and DEC. In FEEC the inner product is found by integrating basis forms (Whitney forms) on simplices [Arnold, Falk, and Winther 2006]:

Definition 2.1.9. Let $\hat{\alpha}$ and $\hat{\beta}$ be the Whitney forms associated to the k -cochains α and β **FEEC Hodge star**, σ_k^{FEEC} , is defined by the formula:

$$\langle \alpha \star_k^{\text{FEEC}} \beta, \sigma^k \rangle = \int \hat{\alpha} \wedge \star \hat{\beta} ,$$

where the Hodge star on the right is the smooth Hodge star.

In DEC the Hodge stars are given by computing the ratio of the volume between the primal and dual cells [Hirani 2003]:

Definition 2.1.10. The **DEC Hodge star**, \star_k^{DEC} is defined by the equation:

$$\frac{1}{|\star \sigma^k|} \langle \star_k^{\text{DEC}} \alpha, \star \sigma^k \rangle := \frac{1}{|\sigma^k|} \langle \alpha, \sigma^k \rangle ,$$

for any k -cochain, α $0 < k < n$. For a 0-cochain, α , \star_0^{DEC} is defined as

$$\frac{1}{s |\star \sigma^0|} \langle \star_0^{\text{DEC}} \alpha, \star \sigma^0 \rangle := \frac{1}{|\sigma^0|} \langle \alpha, \sigma^0 \rangle ,$$

where s is defined by:

$$s = (-1)^{n-1} \operatorname{sgn}(\partial(\star\sigma^0), \star\sigma^1) .$$

Likewise, we define the DEC Hodge star, \star_n^{DEC} , cochain for n -cochains, α :

$$\frac{1}{|\star\sigma^n|} \langle \star_n^{\text{DEC}} \alpha, \star\sigma^n \rangle := \frac{1}{s|\sigma^n|} \langle \alpha, \sigma^n \rangle ,$$

where s is defined by giving σ^{n-1} the orientation induced from σ^n , then if $\star\sigma^{n-1}$ points away from $\star\sigma^n$ then $s = (-1)^{n-1}$ otherwise give it the opposite sign.

While both of these Hodge stars are sparse, the DEC Hodge star is diagonal which has computational advantages. For example the FEEC Hodge star is computationally difficult to invert. Another Hodge star we will consider is the topological Hodge star:

Definition 2.1.11. The **topological Hodge star**, \star_k^{TOP} is defined as an identity matrix, i.e. for any k -cochain α :

$$\langle \star_k^{\text{TOP}} \alpha, \star\sigma^k \rangle := \langle \alpha, \sigma^k \rangle .$$

This Hodge star features prominently in topological field theories such as Chern-Simons which we discuss later in this Chapter and in greater depth in Chapter 6.

In DEC the exterior derivatives and Hodge stars form a complex, that is $d^2 = 0$, as shown in Figure 2.3. The vertical lines are isomorphisms given by the discrete Hodge star.

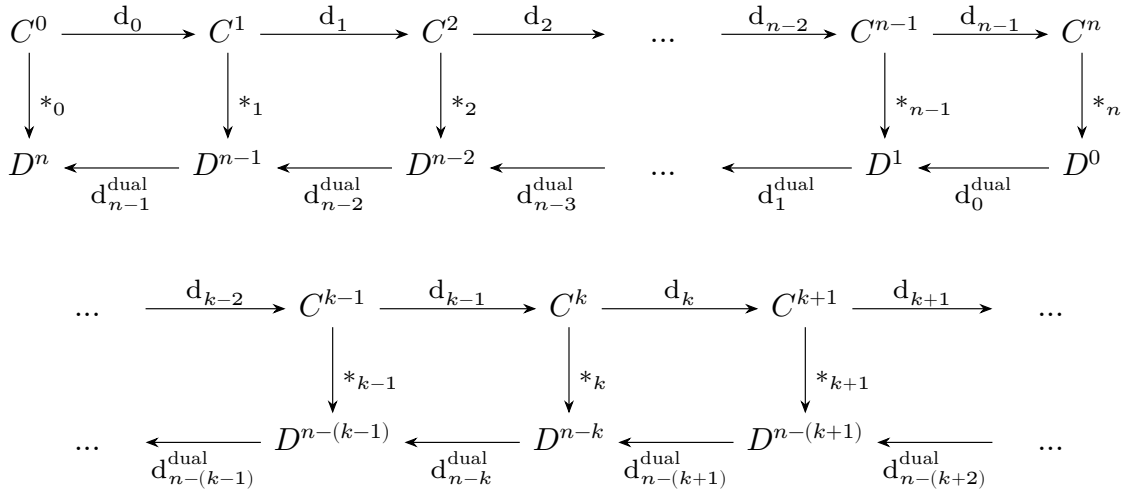


Figure 2.3.: Complex for primal and dual cochains. The top diagram shows the full complex while the second shows a generic rectangle in the diagram.

The Laplacian is a key operator for field theories. Many of equations of motion of the Bosonic theories common in physics (scalar field theory, Yang-Mills, etc) are wave equations for which the Laplacian is the key ingredient. Fortunately the Laplacian is completely defined in terms of operators we have already defined: the discrete exterior derivative and Hodge stars.

Definition 2.1.12. Given α a primal k -cochain, the **discrete primal Laplacian**, Δ_k is given by:

$$\Delta_k \alpha = \left(*_k^{-1} d_{n-k-1}^{\text{dual}} *_k d^k + d^{k-1} *_k^{-1} d_{n-k}^{\text{dual}} *_k \right) \alpha ,$$

and likewise given a dual $(n-k)$ -cochain α^* , the **discrete dual Laplacian** Δ_{n-k}^* is given by:

$$\Delta_k \alpha = \left(*_k d^{k-1} *_k^{-1} d_{n-k}^{\text{dual}} + d_{n-k-1}^{\text{dual}} *_k d^k *_k^{-1} \right) \alpha^* .$$

The **discrete codifferential** is the operator defined by:

$$d_k^* = *_k^{-1} d_{n-k}^{\text{dual}} *_k .$$

In finite element method calculations, mixed weak forms are typically used which involves introducing auxiliary variables and avoids inverses of matrices. However since the DEC Hodge star is diagonal, mixed and direct as well as strong and weak forms are all accessible in calculations in DEC. While we will examine this operator in grater detail in Chapter 3 we want to address two points. The first is that this operator is commonly written in terms of a lower case delta, δ , however we will reserve δ for variations of cochains in Chapters 4 and 5 and so we used the superscript asterisk to denote the codifferential. This comes from the fact that the codifferential is the adjoint to the exterior derivative, a fact we prove in Chapter 3. Also, both the discrete exterior derivative and codifferential commute with the Laplacian. That is:

$$\begin{aligned} d^k \Delta_k &= \Delta_{k+1} d^k \\ d_k^* \Delta_k &= \Delta_{k-1} d_k^* . \end{aligned}$$

This means the eigenfunctions of the k -Laplacian gives insights into the eigenfunctions of the $(k+1)$ and $(k-1)$ -Laplacians. Decomposing the non-harmonic forms through Hodge decomposition we arrive at the following diagram:

2.2. Vector Bundles

In the smooth setting, fiber bundles are defined as a fixed topological space over each point in the base space that is “glued together” properly. The fixed topological space that are “glued” are called *fibers*. For a more formal treatment see [Kobayashi and Nomizu 1996]. We will, however, want to compile some important properties of connections and curvature in the smooth setting now as they will be helpful in our discretization.

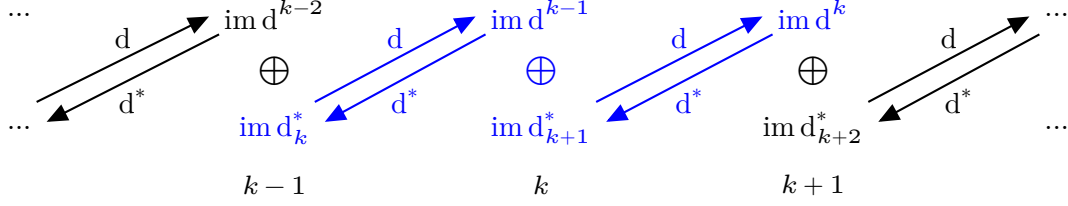


Figure 2.4.: Decomposition of the eigenspaces of the Laplacian. Blue colored parts of this figure show the decomposition for a particular eigenvalue.

We will denote the *total space* of the bundle as E , and the base space as M and use the standard notation $E \rightarrow M$ for a bundle over a base space. When a specific fiber is needed, we will use the notation E_{v_i} to represent the vector space V over the vertex v_i . The dimension of the fiber is called the **rank** of the vector bundle.

Functions from the base space to the vector space are called *sections* and will denoted by s . Sections can be generalized to vector-valued differential forms:

Definition 2.2.1. Given a smooth bundle $E \rightarrow M$ with fiber V , a **vector-valued k -form** is a smooth section of:

$$(V \times M) \otimes \Lambda^k T^*M .$$

Note that a vector-valued k -form, ω can be written as:

$$\omega = v \otimes \alpha ,$$

where $v \in V$ and $\alpha \in \Lambda^k$. Given a path on the base space, we would like to lift this path to the total space. However, since the total space is larger than the base space, this requires a choice. Choosing how to lift any path defines parallel transport. Formally, this is

Definition 2.2.2. Given a vector bundle $E \rightarrow M$ and a path $\gamma : [a, b] \rightarrow M$ **parallel transport** along that path is a linear isomorphism $\phi_\gamma(t_f, t_i) : E_{\gamma(t_i)} \rightarrow E_{\gamma(t_f)}$ such that for any $a \leq t_i < t < t_f \leq b$, $\phi_\gamma(t_f, t_i) = \phi_\gamma(t_f, t) \circ \phi_\gamma(t, t_i)$.

This implies that $\phi_\gamma(t_i, t_f) = (\phi_\gamma(t_f, t_i))^{-1}$. Note that parallel transport is path dependent. That is if γ_1 and γ_2 are two paths such that $\gamma_1(t_i) = \gamma_2(t_i)$ and $\gamma_1(t_f) = \gamma_2(t_f)$ then it is not necessarily true that $\phi_{\gamma_1}(t_f, t_i) = \phi_{\gamma_2}(t_f, t_i)$.

Parallel transport gives us a way to compare elements in different fibers of the vector bundle which we can use to define a derivative called a connection:

Definition 2.2.3. Given a vector bundle with parallel transport and a section s , a **connection** is defined by:

$$(\nabla s)(\gamma(t_i)) := \lim_{t \rightarrow t_i} \frac{\phi_\gamma(t_i, t)s(\gamma(t)) - s(\gamma(t_i))}{t - t_i} .$$

This is also called a **covariant derivative**. **Vector-valued-cochains** as defined as linear combinations of terms of the form:

$$\alpha = v \otimes \omega ,$$

where v is a vector field and ω a differential form. We can extend the connection to a **covariant exterior derivative** by the formula:

$$d^\nabla \alpha := (\nabla v) \wedge + v \otimes (d\omega) .$$

In the case that α is a zero-form, this is equivalent to the connection defined earlier, and so we can denote the connection as either d^∇ or ∇ . Applying the covariant exterior derivative twice gives the curvature, that is:

Definition 2.2.4. Given a vector bundle with connection (V, E, M, ∇) the **curvature**, F , is defined as $F = d^\nabla d^\nabla$ which is also commonly denoted $\nabla \nabla$.

Note that the curvature is a endomorphism-valued two form, that is it acts on sections of the vector bundle and returns a vector valued two-form. In the next subsection recall properties of endomorphism-valued forms more generally, though an important property of curvature is that it is in the kernel of the connection, that is $d^\nabla F = 0$ which is known as the **Bianchi identity**.

We compile some properties of smooth endomorphism-valued forms. These are the properties we will create discrete analogs of in Chapter 5.

Definition 2.2.5. Given a smooth vector bundle $E \rightarrow M$ with fiber V , an endomorphism valued k -form is a smooth section of $(\text{End}(V) \times M) \otimes \Lambda^k T^*M$.

Note that given an endomorphism-valued k -form A , A can be written as:

$$A = \xi \otimes \alpha ,$$

where $\xi \in \text{End}(V)$ and $\alpha \in \Lambda^k T^*M$. In the smooth setting there are a variety of properties these objects have namely:

Proposition 2.2.6. *An endomorphism-valued form, A , acts on a vector-valued form $\omega = v \otimes \beta$ via the equation:*

$$A \wedge \omega = (\xi v) \otimes (\alpha \wedge \beta) ,$$

or linear combinations of such terms.

Proposition 2.2.7. *The covariant exterior derivative does not form a chain complex. Instead applying d^∇ twice to a vector-valued k -form ω gives:*

$$d^\nabla d^\nabla \omega = F \wedge \omega .$$

where F is the curvature.

Definition 2.2.8. The covariant exterior derivative can be extended to endomorphism valued forms through the following formula. Given a endomorphism-valued l -form A and vector-valued k -form ω :

$$(d^\nabla A) \wedge \omega := d^\nabla (A \wedge \omega) - (-1)^k A \wedge d^\nabla \omega .$$

Furthermore, like the exterior derivative there is a codifferential for the covariant exterior derivative:

Definition 2.2.9. The codifferential of the covariant exterior derivative is defined as:

$$(d^\nabla)^* := * d^\nabla * .$$

Note that this operator is the adjoint to the covariant exterior derivative.

2.3. Discrete Vector Bundles

Discrete vector bundles will be described here on simplicial complexes for notational convenience. The definitions and propositions translate to without change to regular cell complexes. In the discrete setting, we do not place a fiber over every point in our simplicial complex, but instead identify a fiber with each vertex.

Definition 2.3.1. Given a simplicial complex K and fixed vector space V a **discrete vector bundle** is an assignment of V to each vertex.

We will refer to the collection of vector spaces as the **discrete total space** E and use the notation borrowed from the smooth setting $E \rightarrow K$ for a vector bundle over the base space K . When a specific fiber is needed, we will use the notation E_{v_i} to represent the vector space V over the vertex v_i . Again, the dimension of the fiber is called the **rank** of the vector bundle.

Definition 2.3.2. A **morphism** between two discrete vector bundles $E \rightarrow K$ and $F \rightarrow L$, is a simplicial map $f : K \rightarrow L$ along with a collection of linear maps: $\{\phi_{v_i} : E_{v_i} \rightarrow F_{f(v_i)}\}$.

Definition 2.3.3. A morphism of vector bundles is called an **isomorphism of vector bundles** if the simplicial map $f : K \rightarrow L$ is a bijection and $\phi_{v_i} : E_{v_i} \rightarrow F_{f(v_i)}$ is a linear isomorphism for all $v_i \in K_0$.

An **automorphism of discrete vector bundles** is an isomorphism where the source and target spaces are the same vector bundle and the simplicial map f on the base is the identity map. Since automorphisms of vector spaces are changes of coordinates, automorphisms of vector bundles are changes of coordinates for each of the fibers.

Definition 2.3.4. A **discrete section** of a discrete vector bundle is a V -valued 0-cochain.

We will use the notation $C^0(K, E)$ to denote the space of discrete sections of the vector bundle $E \rightarrow K$. As with real and complex-valued cochains we will denote dual sections by a superscript asterisk and the space of dual sections by $D^0(\star K, E)$.

As we are free to choose different coordinates on each fiber, we need some way to compare the value of a section in two different fibers. Parallel transport gives us a way to relate the coordinates in one fiber with those to another.

Definition 2.3.5. Given a discrete vector bundle $E \rightarrow K$, **parallel transport** is a linear isomorphism $\langle U, [v_j, v_i] \rangle : E_{v_i} \rightarrow E_{v_j}$ for each edge $[v_i, v_j] = \sigma^1 \in K_1$.

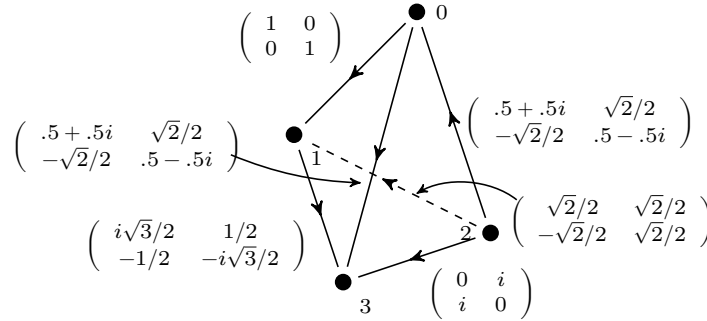


Figure 2.5.: Example of parallel transport with $SU(2)$ structure group. The $SU(2)$ matrices live on the edges and the orientation of the edges that the matrices transport sections.

Evaluation on an edge $[v_i, v_j]$ is written as evaluation on an edge $[v_j, v_i]$ for readability of composition. For example to compose the parallel transports on edges $[v_i, v_j]$ and $[v_j, v_k]$ we would have $\langle U, [v_k, v_j] \rangle \langle U, [v_j, v_i] \rangle$. If ϕ is an automorphism of the vector bundle $E \rightarrow K$ with parallel transport matrices U , the parallel transport is then transformed $\langle U, [v_j, v_i] \rangle \mapsto \langle \phi, [v_j] \rangle \langle U, [v_j, v_i] \rangle \langle \phi^{-1}, [v_i] \rangle$. If we have chosen bases for all of the fibers, we call an automorphism of vector bundles a **gauge transformation**. Two quantities are called **gauge equivalent** if there is a gauge transformation taking one into the other.

Definition 2.3.6. The **structure group** of a vector bundle is a group G such that each parallel transport matrix is gauge equivalent to an element of G .

After choosing bases for all of the fibers E_{v_i} , we can identify each $\langle U, [v_j, v_i] \rangle$ with a matrix in the structure group of the vector bundle. Because of this we often refer the $\langle U, [v_j, v_i] \rangle$ as parallel transport matrices.

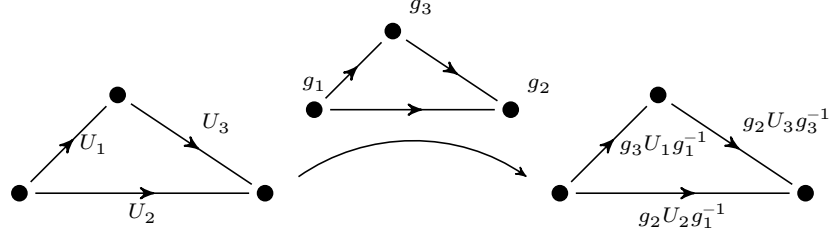


Figure 2.6.: Example of a gauge transformation transforming the parallel transport matrices.

Note that every vector bundle of rank n has a structure group that is a subgroup of the general linear group $GL(n)$. See Chapter 8 for algorithms for finding if a simpler structure group exists for a discrete bundle and, if such a reduction of structure group is possible, how to transform the bundle.

2.4. Observables and the Space of Observables

Physical observables should be gauge-invariant; meaning that under gauge transformations, the number that is measured does not change. Since for any discrete vector bundle, there are N_1 edges that need to be assigned a parallel transport matrix and N_0 vertices on which gauge transformations can act we find the following proposition:

Proposition 2.4.1. *The set of isomorphism classes of discrete vector bundles with structure group G is G^{N_1}/G^{N_0} , where the quotient is by action of the gauge transformations.*

Proof. Recall that a discrete vector bundle is a choice of parallel transport matrix on each of the edges, but two vector bundles are equivalent if there is a gauge transformation that transforms one into the other. \square

Definition 2.4.2. An **observable** is a function $f : G^{N_1}/G^{N_0} \rightarrow \mathbb{R}$.

2.4.1. Holonomy

Holonomy is the measure of how much a vector gets “rotated” as it moves around a loop.

Definition 2.4.3. A **path** γ in a simplicial complex is an ordering of vertices $\{v_{j_i}\}$ such that v_{j_i} and $v_{j_{i+1}}$ share an edge.

Definition 2.4.4. A **loop** is a path of n vertices $\gamma = \{v_{j_i}\}$ such that $v_{j_1} = v_{j_n}$.

Definition 2.4.5. Given a loop γ the **holonomy** around the loop, denoted as $\langle \text{hol}, \gamma \rangle$, is the oriented product of the parallel transport matrices around the loop.

For example, consider the loop $\{1, 2, 3, 1\}$ in Figure 2.5, the holonomy is:

$$\begin{aligned}
& \begin{pmatrix} i\sqrt{3}/2 & 1/2 \\ -1/2 & -i\sqrt{3}/2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} -i\sqrt{3}/2 & -1/2 \\ 1/2 & i\sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-i+\sqrt{3}}{2\sqrt{2}} & \frac{i+\sqrt{3}}{2\sqrt{2}} \\ \frac{i-\sqrt{3}}{2\sqrt{2}} & \frac{i+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
\end{aligned}$$

Given a two cell σ^2 the **loop associated to the two cell** is the loop that begins and ends at the lowest vertex in the two cell, referred to as the **base vertex** of the cell.

Proposition 2.4.6. *Under a gauge transformation $\{\langle g, v_i \rangle\}$ the holonomy around a simplex σ^2 with base vertex v_i transforms as $\langle \text{hol}, \sigma^2 \rangle \mapsto \langle g, [v_i] \rangle \langle \text{hol}, \sigma^2 \rangle \langle g^{-1}, [v_i] \rangle$.*

Proof. Write $\sigma^2 = [v_i, v_j, v_k]$. The holonomy around the simplex is then:

$$\langle \text{hol}, \sigma^2 \rangle = \langle U, [v_i, v_k] \rangle \langle U, [v_k, v_j] \rangle \langle U, [v_j, v_i] \rangle.$$

Under a gauge transformation we have:

$$\begin{aligned}
\langle \text{hol}, \sigma^2 \rangle' &= \langle g, [v_i] \rangle \langle U, [v_i, v_k] \rangle \langle g^{-1}, [v_k] \rangle \langle g, [v_k] \rangle \langle U, [v_k, v_j] \rangle \langle g^{-1}, [v_j] \rangle \\
&\quad \times \langle g, [v_j] \rangle \langle U, [v_j, v_i] \rangle \langle g^{-1}, [v_i] \rangle \\
&= \langle g, [v_i] \rangle \langle U, [v_i, v_k] \rangle \langle U, [v_k, v_j] \rangle \langle U, [v_j, v_i] \rangle \langle g^{-1}, [v_i] \rangle \\
&= \langle g, [v_i] \rangle \langle \text{hol}, \sigma^2 \rangle \langle g^{-1}, [v_i] \rangle.
\end{aligned}$$

□

Proposition 2.4.7. *Under change of base point, $[v_i, v_j, v_k] \mapsto [v_j, v_k, v_i]$ the holonomy around a simplex, σ^2 , transforms as $\langle \text{hol}, [v_i, v_j, v_k] \rangle \mapsto \langle U, [v_i, v_j] \rangle \langle \text{hol}, [v_i, v_j, v_k] \rangle \langle U, [v_j, v_i] \rangle$.*

Proof. The holonomy around the vertex with base point v_i is

$$\langle \text{hol}, [v_i, v_j, v_k] \rangle = \langle U, [v_i, v_k] \rangle \langle U, [v_k, v_j] \rangle \langle U, [v_j, v_i] \rangle.$$

Now consider the cyclic permutation $[v_i, v_j, v_k] \mapsto [v_j, v_k, v_i]$

$$\begin{aligned}
\langle \text{hol}, [v_j, v_k, v_i] \rangle &= \langle U, [v_j, v_i] \rangle \langle U, [v_i, v_k] \rangle \langle U, [v_k, v_j] \rangle \\
&= \langle U, [v_j, v_i] \rangle \langle U, [v_i, v_k] \rangle \langle U, [v_k, v_j] \rangle \langle U, [v_j, v_i] \rangle \langle U, [v_i, v_j] \rangle \\
&= \langle U, [v_j, v_i] \rangle \langle \text{hol}, [v_i, v_j, v_k] \rangle \langle U, [v_i, v_j] \rangle.
\end{aligned}$$

□

Note that this means that trace of holonomy is gauge invariant.

Definition 2.4.8. The trace of the holonomy is called a **Wilson Loop**.

2.5. Connections and Curvature

A choice of parallel transport isomorphisms defines a **connection** on a vector bundle by following definition.

Definition 2.5.1. Given a discrete vector bundle with parallel transport matrices U , and a discrete section s a **discrete connection** is defined by:

$$\langle \nabla s, [v_i, v_j] \rangle := \langle U, [v_i, v_j] \rangle \langle s, [v_j] \rangle - \langle s, [v_i] \rangle .$$

Which we will also refer to as the **discrete covariant derivative of sections**. Other discretizations have also been considered, for example [Leok 2004] obtains a different discretization from the Atiyah exact sequence.

From the Ambrose-Singer Theorem, the curvature is given by the Lie algebra element that generates the holonomy [Ambrose and Singer 1953]. In terms of Taylor expansions this tells us that given the curvature F the holonomy is:

$$\langle \text{hol}, \sigma^2 \rangle = e^{i \int_{\sigma^2} F} ,$$

where we are following the physics convention of defining curvature as a Hermitian operator instead of the math convention of an anti-Hermitian operator. In the infinitesimal limit of the two-cell $\sigma^2 \rightarrow 0$ we can expand to leading order:

$$\langle \text{hol}, \sigma^2 \rangle \approx \mathbb{1} + i \int_{\sigma^2} F ,$$

inspiring the definition of the discrete curvature:

Definition 2.5.2. The **discrete curvature** is given by:

$$\langle F, \sigma^2 \rangle = \langle -i(\text{hol} - \mathbb{1}), \sigma^2 \rangle .$$

This equation is common in the lattice gauge theory literature [Creutz 1986] and was used to extend FEEC to discretize Yang-Mills [Christiansen and Halvorsen 2011]. In Sections 5.1.1 and 5.1.2 we discuss extensions of the discrete connection to general vector-valued cochains to derive the discrete curvature from formulas inspired by the smooth formula: $F = \nabla \nabla$.

2.6. Characteristic Classes

Given a complex vector bundle $E \rightarrow M$ the k -th Chern class is an element of $H^{2k}(M, \mathbb{Z})$ that obeys the following axioms :

1. Naturality: If $f : M \rightarrow N$ is a map and $E \rightarrow N$ is a vector bundle then $[c_k(f^*E)] = f^*[c_k(E)]$.
2. Additivity: If $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is an exact sequence of vector bundles, then $[c(E)] = [c(E')] \smile [c(E'')]$.
3. Normalization: If E is a line bundle then $[c(E)] = 1 + [e(E_R)]$, where $e(E_R)$ is the Euler class of the underlying real two-dimensional vector bundle.

where $[c(E)] = \sum_k [c_k(E)]$ is called the *total Chern class*. In terms of Chern-Weil theory the Chern classes can be written in terms of curvature by [Roe 1999]:

$$\det \left(\mathbf{1} + \frac{iFt}{2\pi} \right) = \sum_k c_k t^k . \quad (2.1)$$

Explicitly, the first three Chern classes are then given by:

$$\begin{aligned} c_0 &= 1 \\ c_1 &= \frac{1}{2\pi i} \operatorname{tr} F \\ c_2 &= \frac{1}{8\pi^2} [\operatorname{tr} F^2 - (\operatorname{tr} F)^2] \\ c_3 &= \frac{i}{48\pi^3} [-2 \operatorname{tr} F^3 + 3 \operatorname{tr}(F^2) \operatorname{tr} F - (\operatorname{tr} F)^3] \end{aligned}$$

where multiplication is defined by wedge product.

Due to axiom 2, if a vector bundle is a direct sum of line bundles, the first Chern class of the line bundles determines all of the remaining Chern classes. The *splitting principle* describes how to construct a space such that any vector bundle splits as sum of line bundles [Roe 1999].

It is also helpful to define the Chern character which is simpler to state in terms of curvature.

Definition 2.6.1. Given a vector bundle with curvature F , k -th **Chern character**, $\operatorname{ch}_k(E)$ is given by $\operatorname{ch}_k(E) = \operatorname{tr} F^k$ for all $k > 0$ and $\operatorname{ch}_0(E) = \mathbf{1}$ for $k = 0$, where multiplication is defined with the wedge product.

In analogy to the total Chern class the **total Chern character** is the formal sum:

$$\operatorname{ch}(E) = \sum_k \operatorname{ch}_k(E) .$$

The Chern character has the following useful properties:

Proposition. Given two vector bundles $E_1 \rightarrow M$ and $E_2 \rightarrow M$ of rank m_1 and m_2 , respectively:

1. $\text{ch}(E_1 \oplus E_2) = \text{ch}(E_1) + \text{ch}(E_2)$,
2. $\text{ch}(E_1 \otimes E_2) = m_2 \text{ch}(E_1) + m_1 \text{ch}(E_2)$.

Comparing Equations (2.1) to the definition of Chern character implies that one can be used to determine the other.

2.7. Discrete BF Models

BF models derive their name from the background field B . These models are topological field theories whose action takes the form:

$$S = \int K[B \wedge F] ,$$

where F is the curvature, B is an endomorphism-valued degree $n-2$ form, and K is a non-degenerate bilinear form. We will be most interested in the abelian case where K can be taken to be simple multiplication.

[Sen et al. 2000]. have a fully-discrete space-time theory on a simplicial complex. As previously noted it requires gauge field doubling, but they are able to demonstrate that their discrete action recovers the topological information of the smooth partition function [Sen et al. 2000]. The action is given by

$$S_{\text{Sen et al.}} = \frac{k}{4\pi} (A^T, (A^*)^T) \begin{pmatrix} 0 & *d \\ *d & 0 \end{pmatrix} \begin{pmatrix} A \\ A^* \end{pmatrix}$$

where gauge fields are placed on both primal and dual edges. The exterior derivative are the usual DEC exterior derivatives on primal and dual co-chains. This theory is an example of using topological Hodge stars, where they are simply the identity map linking primal n -cochains with dual $(n-k)$ -cochains. An alternative discretization of BF is provided by Pavel Mnev [Mnev 2007], which instead of discretization the gauge fields focuses on discretizing the algebraic properties of BF theory.

Chapter 3.

Discrete Exterior Calculus with Boundary

In this Chapter, we extend the discrete exterior derivative on dual cochains to account for the boundary which requires additional dual cells to be added on the boundary of our discretized space. We show that this exterior derivative leads to a codifferential which is the adjoint (up to the usual boundary terms) of the discrete exterior derivative.

3.1. Exterior Derivative

In the Preliminaries, we defined, for complexes without boundary, the discrete exterior derivative acting on dual $(n - k)$ cochains in terms of the primal discrete exterior derivative as:

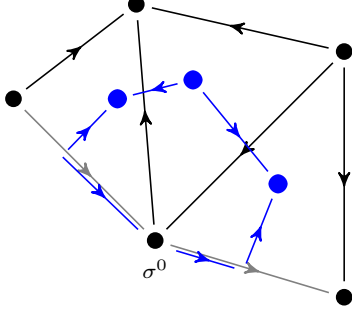
$$d_{n-k}^{\text{dual}} = (-1)^k \left(d^{k-1} \right)^T . \quad (3.1)$$

This definition, however must be extended to account for spaces with boundary. The boundary of a dual cell that touches the complex boundary must include cells that are not part of the original dual complex as shown in Figure 3.1.

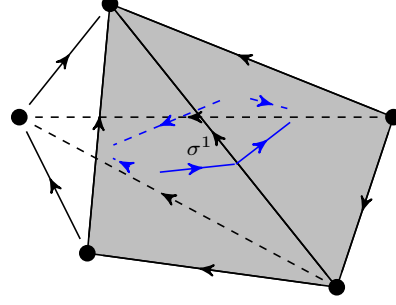
We will call these added cells **boundary duals** and these are constructed by forming the dual mesh to the boundary complex where duality on the boundary is with respect to the boundary which is one dimension lower. We will call the original dual cells for our complex **interior duals**. With this addition the dual exterior derivative is given as the adjoint to the dual operator on dual chains and one can incorporate boundary conditions more generally. Recall that as discussed in Chapter 2, both the boundary and the top-dimensional cells are oriented and so the boundary dual mesh can be created as described in [Hirani 2003]. As was the case of a complex without boundary this can be related to the dual exterior derivative.

Definition 3.1.1. The **dual discrete exterior derivative** is defined as:

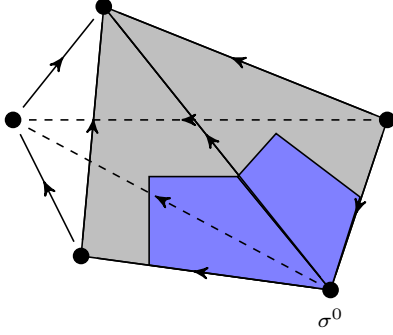
$$d_{n-k}^{\text{dual}} = \left[(-1)^k d_{k-1}^T, (-1)^{k-1} i_{k-1}^\partial \right] .$$



(a) Boundary of a dual two-cell, $\star\sigma^0$, in two dimensions.



(b) Boundary to a dual two-cell, $\star\sigma^1$, in three dimensions.



(c) Boundary to a dual three-cell, $\star\sigma^0$, in three dimensions.

Figure 3.1.: Examples of the boundary mesh in two (a) and three (b) & (c) dimensions. The black lines denote the original primal mesh and the lighter gray cells denote the boundary of that mesh. The blue cells are the interior duals and the light blue cells are boundary duals.

where i_{k-1}^∂ is the additional entries that come from the boundary dual cells. Explicitly, i_{k-1}^∂ is a $N_{k-1} \times N_{k-1}^\partial$ matrix with a zero row for each $(k-1)$ -cell that is not on the boundary. For the cells on the boundary, the values in the row are all 1, for each column that represents the boundary dual cell of the primal boundary cell. For a dual $(n-k)$ -cochain α^* we will write the application of d_{n-k}^{dual} to α^* as:

$$d_{n-k}^{\text{dual}} \alpha^* = \left[(-1)^k d_{k-1}^T \alpha^*, (-1)^{k-1} i_{k-1}^\partial \alpha^{*\partial} \right],$$

where on the right α^* is the part of the dual cochain on volumetric dual and $\alpha^{*\partial}$ denotes the boundary dual part. For the complexes shown in Figure 3.1, we have the following d_{n-k}^{dual} :

Example 3.1.2. Consider the boundary dual cell whose outline is shown in Figure 3.1a. We can write the boundary operator for this term as the following matrix:

$$\partial \star \sigma^0 = [1, -1, 1, -1, 1, 1].$$

Here we have traveled counter-clockwise around the loop (following the orientation of the space), beginning with the interior dual half edge on the right. This means that the last two elements are the ones that corresponds to the inclusion map i_0^∂ .

Example 3.1.3. Consider the boundary dual cell whose outline is shown in Figure 3.1b. In many ways this example is parallel to the previous. We can again derive the boundary operator for this dual two cell and represent it as the following matrix:

$$\partial \star \sigma^1 = [-1, 1, -, 1, 1, 1, 1] .$$

Again we traveled in a counter-clockwise direction as viewed from above and the boundary edges are represented by the last two elements in the operator, which becomes the basis for the inclusion map i_1^∂ .

Example 3.1.4. Consider the boundary dual cell whose outline is shown in Figure 3.1c. Unlike our previous examples this dual region is a three-dimensional volume with two-dimensional boundary, although the strategy is the same. First we derive the boundary operator as a matrix:

$$\partial \star \sigma^0 = [1, -1, 1, 1, 1] .$$

The last two elements are the terms that comes from the boundary dual two cell which is the term that transforms into the inclusion map i_0^∂ .

Recall that the most important property of the exterior derivative is that acting twice on any form with it yields zero. Indeed that is the case for our discrete dual exterior derivative:

Proposition 3.1.5.

$$d_{n-k-1}^{dual} d_{n-k}^{dual} = 0$$

Proof. Let α^* be a dual $(n-k)$ -cochain, we then have:

$$\begin{aligned} \langle d_{n-k-1}^{dual} d_{n-k}^{dual} \alpha^*, \star \sigma^{k-2} \rangle &= \langle d_{n-k}^{dual} \alpha^*, \partial \star \sigma^{k-1} \rangle \\ &= \langle \alpha^*, \partial \partial \star \sigma^{k-1} \rangle \\ &= 0 . \end{aligned}$$

□

We should however note that the above proposition was also true for the original definition of the dual exterior derivative given in Chapter 2. And indeed in the case of boundaryless CW complexes the two definitions are equivalent. However, our extended definition is able to reproduce discrete analogs of boundary terms that arise in the smooth setting that the original definition does not, which is the topic of the following section.

3.2. Codifferential and Adjointness

The codifferential is the formal adjoint under the inner product of differential forms. In the smooth setting, the codifferential applied to k -forms is defined as:

$$d^* = (-1)^{k(n-k)} * d *$$

Usually δ is used for the codifferential, however we need to reserve δ to take variations in the next chapter. Note that d^* is sometimes used for the formal adjoint of d . Our discrete codifferential takes inspiration from the above formula, however given a primal k -cochain α the value of $*_k \alpha$ on the boundary duals is given by the boundary condition $\text{tr} * \alpha$ and does not come from the values of α evaluated on the primal cells. That is, we will use $\text{tr} * \alpha$ notation for the resulting discrete object which is a boundary dual cochain in D_{∂}^{k-1} . When considering which dual cells on the boundary the dual cochain $\text{tr} * \alpha$ resides, the dimension of such cells is determined by the degree of $* \alpha$. For example for $n = 3$ and α a primal 1-cochain, $* \alpha$ is a dual 2-cochain and hence resides on the cells that are boundary duals of the primal vertices on the boundary, which are 2-dimensional patches.

Remark 3.2.1. The dual of a primal k -cochain is a dual $(n - k)$ -cochain in the interior. On the boundary the corresponding dual object is the dual of a primal $(k - 1)$ -cochain (duality here is with respect to the boundary dimension, $n - 1$) since $(n - 1) - (k - 1) = n - k$. We will refer to the space of these cochains as D_{∂}^{k-1} . The notation $\text{tr} * \alpha$ will produce such an object on the boundary starting with a primal k -cochain α .

Definition 3.2.2. Given a cellular complex with boundary K and a primal k -cochain, α , the **discrete codifferential** is given by:

$$d_k^* \alpha := (-1)^{n(k-1)+1} *_k^{-1} d_{n-k}^{\text{dual}} \begin{bmatrix} *_k \alpha \\ \text{tr} * \alpha \end{bmatrix}.$$

Proposition 3.2.3. *The discrete codifferential is the adjoint to the discrete exterior derivative. That is:*

$$(d^{k-1} \beta, \alpha) - (\beta, d_k^* \alpha) = \beta^T i_{k-1}^{\partial} \text{tr} * \alpha,$$

for any primal k -cochain α and primal $(k - 1)$ -cochain β .

Proof. Consider the inner product:

$$\begin{aligned} (\beta, d_k^* \alpha) &= (\beta, (-1)^{n(k-1)+1} *_k^{-1} ((-1)^k d_{k-1}^T *_k \alpha + (-1)^{k-1} i_{k-1}^{\partial} \text{tr} * \alpha)) \\ &= (-1)^{nk-n+1+k} \beta^T *_k^{-1} (-1)^k *_k^{-1} (d_{k-1}^T *_k \alpha - i_{k-1}^{\partial} \text{tr} * \alpha) \\ &= (-1)^{nk-n+1+k+(k-1)(k-1)} (\beta^T d_{k-1}^T *_k \alpha - \beta^T i_{k-1}^{\partial} \text{tr} * \alpha) \\ &= (d^{k-1} \beta, \alpha) - \beta^T i_{k-1}^{\partial} \text{tr} * \alpha. \end{aligned}$$

□

We can also define a codifferential for the dual exterior derivative. Again we take inspiration from the smooth setting by defining the dual codifferential as follows.

Definition 3.2.4. The discrete codifferential on a dual $(n - k)$ -cochain α^* is:

$$\left(d_{n-k}^{\text{dual}}\right)^* \alpha^* = (-1)^{(k+1)(n-k)} *_k \text{d}_k *_k^{-1} \alpha^* .$$

Proposition 3.2.5. *The dual discrete exterior derivative and dual codifferential are adjoints. That is, given a dual $(n - k)$ -cochain α^* and dual $(n - k - 1)$ -cochain β^* :*

$$\left(d_{n-k-1}^{\text{dual}} \beta^*, \alpha^*\right) - \left(\beta^*, \left(d_{n-k}^{\text{dual}}\right)^* \alpha^*\right) = (-1)^k \left(i_k^{\partial} \beta^* \partial\right)^T *_k^{-1} \alpha^* .$$

Proof.

$$\begin{aligned} \left(d_{n-k-1}^{\text{dual}} \beta^*, \alpha^*\right) &= \left((-1)^{k+1} \text{d}_k^T \beta^* + (-1)^k i_k^{\partial} \beta^* \partial\right)^T *_k^{-1} \alpha^* \\ &= (\beta^*)^T (-1)^{k+1} \text{d}_k *_k^{-1} \alpha^* + \left(i_k^{\partial} \beta^* \partial\right)^T (-1)^k *_k^{-1} \alpha^* \\ &= (\beta^*)^T (-1)^{k+1+(k+1)(n-k-1)} *_k^{-1} *_k \text{d}_k *_k^{-1} \alpha^* + \left(i_k^{\partial} \beta^* \partial\right)^T (-1)^k *_k^{-1} \alpha^* \\ &= (\beta^*)^T *_k^{-1} (-1)^{(k+1)(n-k)} *_k \text{d}_k *_k^{-1} \alpha^* + \left(i_k^{\partial} \beta^* \partial\right)^T (-1)^k *_k^{-1} \alpha^* \\ &= (\beta^*)^T *_k \left(d_{n-k}^{\text{dual}}\right)^* \alpha^* + \left(i_k^{\partial} \beta^* \partial\right)^T (-1)^k *_k^{-1} \alpha^* \\ &= \left(\beta^*, \left(d_{n-k}^{\text{dual}}\right)^* \alpha^*\right) + \left(i_k^{\partial} \beta^* \partial\right)^T (-1)^k *_k^{-1} \alpha^* . \end{aligned}$$

□

3.3. Primal-Dual Wedge Product & Discrete Exterior Derivative

We provide a rule for orienting the convex hull of a simplex and its dual based on the relative orientations of the two simplices. We then use this to enhance the definition of the top-dimensional primal-dual wedge product given in Chapter 2 to account for this orientation. This orientation is important because with it the top-dimensional primal-dual wedge product can be related to the inner products we have been working with this chapter and because of this a discrete version of integration by parts is proved. We also define a co-dimension one primal-dual wedge product and prove a Stokes' theorem for this.

Recall from the preliminary that given a primal and dual cell complex of dimension n there is a natural wedge product between k -cochains and dual $(n - k)$ -cochains:

$$\langle \alpha \wedge \beta^*, V_{\sigma^k} \rangle = \frac{1}{n} \langle \alpha, \sigma^k \rangle \langle \beta^*, \star \sigma^k \rangle .$$

This definition will always give a top-dimensional form which will be defined on the convex hull

associated to the primal k -cell and dual $(n - k)$ -cell, denoted V_{σ^k} . However, this definition does not account for the orientation of the convex hull which is inherited from the orientation of the cell complex. In a two-dimensional space, for a zero-cell V_{σ^0} is the same as $\star\sigma^0$ and has the same orientation, V_{σ^2} is the same as σ^2 and has the same orientation, V_{σ^1} is oriented in agreement with the ambient space, while $V_{\star\sigma^1}$ is oriented oppositely as shown in Figure 3.2 and explained in general below.

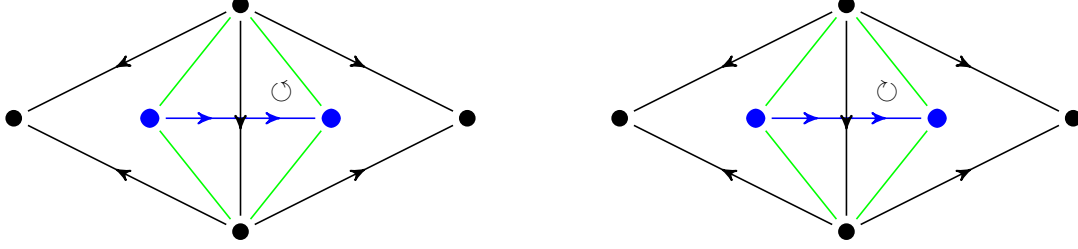


Figure 3.2.: V_{σ^1} and $V_{\star\sigma^1}$ in two-dimensions. Note that they have opposite orientation from V_{σ^1} and $V_{\star\sigma^0}$.

Generally, V_{σ^k} is oriented in agreement with the ambient space, while $V_{\star\sigma^k}$ may not be. We determine the orientation of $V_{\star\sigma^k}$ by thinking about the vectors that span σ^k (call them $\{v_1, v_2, \dots, v_k\}$) and $\star\sigma^k$ (call them $\{v_k, v_{k+1}, \dots, v_n\}$). The ambient space is oriented according to $\{v_1, v_2, \dots, v_n\}$ and $V_{\star\sigma^k}$ is oriented according to the permutation $\{v_{k+1}, \dots, v_n, v_1, v_2, \dots, v_k\}$. Effectively, this means $V_{\star\sigma^k}$ has opposite orientation to V_{σ^k} when $k(n - k)$ is odd.

However this definition does not account for the orientations of the cells, which can be corrected by the following enhanced definition:

Definition 3.3.1. Given a primal k -cochain α and a dual $(n - k)$ -cochain β^* the **top-dimensional primal-dual wedge product** is given by:

$$\begin{aligned} \langle \alpha \wedge \beta^*, V_{\sigma^k} \rangle &:= \frac{1}{n} \langle \alpha, \sigma^k \rangle \langle \beta^*, \star\sigma^k \rangle \\ \langle \beta^* \wedge \alpha, V_{\star\sigma^k} \rangle &:= \frac{1}{n} (-1)^{k(n-k)} \langle \beta^*, \star\sigma^k \rangle \langle \alpha, \sigma^k \rangle, \end{aligned}$$

where the second definition accounts for the orientation of $V_{\star\sigma^k}$ described above.

The top-dimensional primal-dual wedge is related to the inner product we have been discussing through the following formula:

$$\sum_{\sigma^k} \langle \alpha \wedge \star\beta, V_{\sigma^k} \rangle = (\alpha, \beta)$$

This formula mirrors the smooth formula that defines the hodge star given in the Preliminaries:

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mu.$$

Note that the volume form does not appear in our discrete formula, this is because the cochains α and β are already integrated. Furthermore due to this relation the primal-dual wedge product has a Leibniz property:

Proposition 3.3.2. *The primal-dual wedge product has a Leibniz-like formula. Given a primal k -cochain α and a dual $(n - k - 1)$ -cochain β^* we have:*

$$\sum_{\sigma^k} \langle d^k \alpha \wedge \beta^*, V_{\sigma^k} \rangle + \langle \alpha \wedge d_{n-k-1}^{\text{dual}} \beta^*, V_{\sigma^k} \rangle = \alpha^T i_k^{\partial} \beta^{*\partial},$$

where $\beta^{*\partial}$ are the values of β^* on the boundary duals.

Proof.

$$\begin{aligned} \sum_{\sigma^k} \langle (d^k \alpha) \wedge \beta^*, V_{\sigma^{k+1}} \rangle &= \sum_{\sigma^k} \langle (d^k \alpha) \wedge *_{k+1} *_{k+1}^{-1} \beta^*, V_{\sigma^{k+1}} \rangle \\ &= (d^k \alpha, *_{k+1}^{-1} \beta^*) , \end{aligned}$$

and likewise:

$$\begin{aligned} \sum_{\sigma^k} \langle \alpha \wedge (d_{n-k}^{\text{dual}} \beta^*), V_{\sigma^{k+1}} \rangle &= \sum_{\sigma^k} \langle \alpha \wedge *_{k+1} *_{k+1}^{-1} (d_{n-k-1}^{\text{dual}} *_{k+1} *_{k+1}^{-1} \beta^*), V_{\sigma^{k+1}} \rangle \\ &= (\alpha, d_{k+1}^* *_{k+1}^{-1} \beta^*) , \end{aligned}$$

Putting this together we have:

$$\begin{aligned} \sum_{\sigma^k} \langle d^k \alpha \wedge \beta, V_{\sigma^k} \rangle + \langle \alpha \wedge d_{n-k-1}^{\text{dual}} \beta^*, V_{\sigma^k} \rangle &= (d^k \alpha, *_{k+1}^{-1} \beta^*) + (\alpha, d_{k+1}^* *_{k+1}^{-1} \beta^*) \\ &= \alpha^T i_k^{\partial} \beta^{*\partial} . \end{aligned}$$

By Proposition 3.2.3. □

We now extend the top-dimensional primal-dual wedge product to a co-dimension one primal-dual wedge product. Important to this definition is that restricted to the boundary (which is of co-dimension one) the co-dimension one wedge product should be equivalent to the primal-dual wedge product we defined previously.

Definition 3.3.3. Let K be a regular CW complex, the **co-dimension one primal-dual wedge product** is defined as follows. For a primal 0-cochain, α , and a dual $(n - 1)$ -cochain, β^* , and an interior dual $(n - 1)$ -chain $\star \sigma^1$ such that σ_1^0 and σ_2^0 are the endpoints of σ^1 :

$$\langle \alpha \wedge \beta^*, \star \sigma^1 \rangle := \left(\frac{\langle \alpha, \sigma_2^0 \rangle + \langle \alpha, \sigma_1^0 \rangle}{2} \right) \langle \beta^*, \star \sigma^1 \rangle .$$

and for a boundary dual $(n-1)$ -chain, $\star\sigma^0 \cap \partial K$:

$$\langle \alpha \wedge \beta^*, \star\sigma^0 \cap \partial K \rangle = \langle \alpha, \sigma^0 \rangle \langle \beta^*, \star\sigma^0 \cap \partial K \rangle .$$

Likewise we can define the co-dimension one primal-dual wedge product between a primal $(n-1)$ -cochain, α and dual 0-cochain β^* . For primal $(n-1)$ -chains not on the boundary we have:

$$\langle \alpha \wedge \beta^*, \sigma^{n-1} \rangle := \langle \alpha, \sigma^{n-1} \rangle \left(\frac{\langle \beta^*, \star\sigma_0^n \rangle + \langle \beta^*, \star\sigma_1^n \rangle}{2} \right) ,$$

where $\star\sigma_0^n$ and $\star\sigma_1^n$ are the endpoints of $\star\sigma^{n-1}$. For primal $(n-1)$ -chains on the boundary we need to only multiply the primal cochain, α with $\beta^{*\partial}$ evaluated on σ^{n-1} 's boundary dual:

$$\langle \alpha \wedge \beta^{*\partial}, \sigma^{n-1} \rangle := \langle \alpha, \sigma^{n-1} \rangle \langle \beta^{*\partial}, \star\sigma^{n-1} \cap \partial K \rangle .$$

The above definition of the co-dimension one primal-dual wedge product allows us to derive a discrete Stokes' Theorem. The left-hand side of the formula is a discrete integral over the volume of the space and the right-hand side is the boundary integral.

Proposition 3.3.4. *Given a primal 0-cochain, α , and dual $(n-1)$ -cochain, β^* , the exterior derivative of the wedge product is equal to:*

$$\sum_{\sigma^0} \langle d_{n-1}^{\text{dual}} (\alpha \wedge \beta^*), \star\sigma^0 \rangle = \alpha^T i_0^\partial \beta^{*\partial} .$$

Similarly, given a primal $(n-1)$ -cochain α and dual 0-cochain β^ , the exterior derivative of the wedge product is equal to:*

$$\sum_{\sigma^n} \langle d^{n-1} (\alpha \wedge \beta^*), \sigma^n \rangle = (i_{n-1}^\partial \beta^{*\partial})^T \alpha$$

Proof. The proofs for these two cases are similar, though different enough to warrant both to be given. First we give the proof for the primal 0-cochain and dual $(n-1)$ cochain. By definition of the dual discrete exterior derivative:

$$\langle d_{n-1}^{\text{dual}} (\alpha \wedge \beta^*), \star\sigma^0 \rangle = \langle \alpha \wedge \beta^*, \partial \star \sigma^0 \rangle .$$

If σ^0 is interior to the domain, boundary of $\star\sigma^0$ is made entirely of interior dual $(n-1)$ -cells, if σ^0 is on the boundary of the domain, the boundary of $\star\sigma^0$ has both interior and boundary dual cells. Each interior dual $(n-1)$ -cell is in the boundary of exactly two $\star\sigma^0$. Since these occur with opposite orientations, these terms cancel. That means we have:

$$\sum_{\sigma^0} \langle d_{n-1}^{\text{dual}} (\alpha \wedge \beta^*), \star\sigma^0 \rangle = \sum_{\sigma^0} \langle (\alpha \wedge \beta^*), \partial \star \sigma^0 \rangle$$

$$\begin{aligned}
&= \sum_{\sigma^0 \in \partial K} \langle \alpha \wedge \beta^{*\partial}, \star \sigma^0 \cap \partial K \rangle \\
&= \alpha^T i_0^\partial \beta^{*\partial} .
\end{aligned}$$

Similarly, the discrete exterior derivative of primal $(n-1)$ -cochain α and dual 0-cochain β^* is defined as:

$$\langle d^{n-1}(\alpha \wedge \beta^*), \sigma^n \rangle = \langle \alpha \wedge \beta^*, \partial \sigma^n \rangle .$$

Again there are two cases. If σ^n is in the interior of the domain, all of the $(n-1)$ -cells that form the faces of σ^n are interior to the domain. If σ^n intersects the boundary of the domain then there are two types of $(n-1)$ -cells that form the faces of σ^n : the one on the boundary of the domain and those in the interior. When summing over all σ^n , the interior faces occur twice with opposite orientations and therefore cancel, leaving only the boundary faces:

$$\begin{aligned}
\sum_{\sigma^n} \langle d^{n-1}(\alpha \wedge \beta^*), \sigma^n \rangle &= \sum_{\sigma^n} \langle \alpha \wedge \beta^*, \partial \sigma^n \rangle \\
&= \sum_{\sigma^{n-1} \in \partial K} \langle \alpha \wedge \beta^*, \sigma^{n-1} \rangle \\
&= (i_{n-1}^\partial \beta^{*\partial})^T \alpha .
\end{aligned}$$

□

In three dimensions α would be a primal 2-cochain and β^* a dual 0-cochain. $\beta^{*\partial}$ is β^* restricted to the boundary of the space and i_2^∂ includes the boundary dual 0-cells into the space of primal 2-cells.

Remark 3.3.5. Generalizing the above Proposition to general forms would require an exterior derivative on the “diamond regions” that arise from primal-dual wedge products.

Chapter 4.

Noether's Theorem for Discrete Field Theories

In this chapter we develop two of the most essential tools for the study of field theories: the discrete analogs of the Euler-Lagrange equations and Noether's first theorem. The key ingredient in the proofs of both of these is the adjointness of the discrete exterior derivative and codifferential which we proved in Chapter 3.

4.1. Fully-Discrete Field Theories

We begin by working with a fully-discrete theory, that is a theory in which both space and time are discretized. In our language these theories are somewhat more compact to write out because the time components of the discretized differential forms do not need to be treated separately. We begin by defining a general discrete action, S .

Given a discrete field theory with primal cochains α_i each of degree k_i and dual cochains β_j^* each of degree $(n - k_j)$, S can be constructed from linear combinations of inner products of α_i , β_j^* , $d^{k_i} \alpha_i$, $d_{n-k_j}^{\text{dual}} \beta_j^*$, along with the necessary hodge stars. Although there are many possible terms in this discrete action, in most physical theories only a few play a role. For instance we will consider the complex scalar field theory, $O(n)$ free field theory (i.e. with a trivial connection), and abelian Yang-Mills (electrodynamics).

Example 4.1.1. Complex scalar field theory is defined by the action:

$$S = \frac{1}{2} (d^0 \phi, d^0 \bar{\phi}) + \frac{m}{2} (\phi, \bar{\phi}) ,$$

where ϕ is a complex-valued primal zero-cochain and $\bar{\phi}$ is its complex-conjugate. Since ϕ is complex it is actually made of two independent, real components $\phi = \phi_1 + i\phi_2$, which we will need to later allow to vary independently. However, it is more common to instead think of ϕ and $\bar{\phi}$ as being the independent variables. This amounts to a change of basis from the $\{\phi_1, \phi_2\}$ basis to the $\{\phi, \bar{\phi}\}$.

Example 4.1.2. The $O(n)$ field theory is similar with a real n -dimensional vector replacing the complex scalar field on each zero-cell. The action is defined by:

$$S = \frac{1}{2} (d^0 \phi, d^0 \phi) + \frac{m}{2} (\phi, \phi) ,$$

where ϕ are real n -dimensional vectors on each zero-cell. Here d^0 is the usual d^0 applied to every entry in the vector and the inner product is formed by taking the transpose of the first term both in the vector dimensions and in the cochain dimensions. Also note that each ϕ has in this case n independent components that will need to be varied independently, as in the complex scalar case. In fact, the complex scalar theory is isomorphic to the $O(2)$ theory.

Example 4.1.3. Abelian Yang-Mills theory is defined by the action:

$$S = \frac{1}{2} (d^1 A, d^1 A)$$

where A is a real one-cochain. For a discussion of Yang-Mills more generally see Chapter 5.

4.1.1. Euler-Lagrange Equations

For classical field theories, the Euler-Lagrange equations describe the equations of motion for the system. Solutions to these equations are also the dominant terms in quantum field theories as they have the highest weighting in the Boltzmann factor [Peskin and Schroeder 1995]. During our derivations we will need a **functional derivative**, which is defined by the Gâteaux derivative. That is, given an arbitrary discrete functional $F[\phi]$, the functional derivative is

$$\left(\frac{\delta F}{\delta \phi}, \eta \right) := \lim_{\epsilon \rightarrow 0} \frac{F[\phi + \epsilon \eta] - F[\phi]}{\epsilon} .$$

To derive the Euler-Lagrange equations we will need to vary the action with respect to the perturbations of the fields:

$$\alpha_i \mapsto \alpha_i + \delta \alpha_i \quad \text{and} \quad \beta_j^* \mapsto \beta_j^* + \delta \beta_j^* ,$$

which induces the variations on the derivatives:

$$d^{k_i} \alpha_i \mapsto d^{k_i} \alpha_i + d^{k_i} \delta \alpha_i \quad \text{and} \quad d_{n-k_j}^{\text{dual}} \beta_j^* \mapsto d_{n-k_j}^{\text{dual}} \beta_j^* + d_{n-k_j}^{\text{dual}} \delta \beta_j^* .$$

This imposes variations on S :

$$\begin{aligned} \delta S = & \sum_{k=0}^n \left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \\ & + \left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) . \end{aligned}$$

And so for the above variation of the action we have:

$$\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) := \lim_{\epsilon \rightarrow 0} \frac{S[\alpha_i + \epsilon \delta \alpha_i, \beta_j^*, d^k \alpha_i, d_{n-k_j}^{\text{dual}} \beta_j^*] - S[\alpha_i, \beta_j^*, d^k \alpha_i, d_{n-k_j}^{\text{dual}} \beta_j^*]}{\epsilon}.$$

Similarly, $\frac{\delta S}{\delta d^{k_i} \alpha_i}$ means take the functional derivative with respect to the slot with $d^{k_i} \alpha_i$. The result of various functional derivatives are most easily seen by example. For instance, when varying the action from complex scalar field theory with respect to variations of ϕ we have:

$$\left(\frac{\delta S}{\delta \phi}, \eta \right) = \frac{m}{2} \lim_{\epsilon \rightarrow 0} \frac{(\phi + \epsilon \eta, \bar{\phi}) - (\phi, \bar{\phi})}{\epsilon},$$

where η is an arbitrary zero-cochain. Recall that we are treating ϕ and $\bar{\phi}$ as independent and so variation of ϕ do not induce variations of $\bar{\phi}$, see Example 4.1.1. Expanding and subtracting like terms gives:

$$\begin{aligned} \left(\frac{\delta S}{\delta \phi}, \eta \right) &= \frac{m}{2} \lim_{\epsilon \rightarrow 0} \frac{(\phi, \bar{\phi}) + \epsilon (\eta, \bar{\phi}) - (\phi, \bar{\phi})}{\epsilon} \\ &= \frac{m}{2} \eta^T *_0 \bar{\phi}. \end{aligned}$$

Identifying like terms we have that:

$$\frac{\delta S}{\delta \phi} = \frac{m}{2} \bar{\phi}.$$

We will now derive the Euler-Lagrange equations as stationary points of the action. We first vary the action with respect to perturbations of the fields. Then we apply adjointness of d and d^* . Finally, we need to regroup terms. We begin by varying the action with respect to perturbations

$$\alpha_i \mapsto \alpha_i + \delta \alpha_i \quad \text{and} \quad \beta_j^* \mapsto \beta_j^* + \delta \beta_j^*,$$

and find that:

$$\begin{aligned} \delta S &= \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \right] \\ &\quad + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) \right]. \end{aligned}$$

We can now apply adjointness of d and d^* , Propositions 3.2.3 and 3.2.5. As usual we will not allow our fields to vary on the boundary and so no boundary terms will appear.

$$\delta S = \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) \right]$$

$$+ \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left((d_{n-k_j-1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) \right] .$$

Combining the terms gives:

$$\begin{aligned} \delta S = \sum_i & \left[\left(\frac{\delta S}{\delta \alpha_i} + d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) \right] \\ & + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*} + (d_{n-k_j-1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) \right] . \end{aligned}$$

Since this must be zero for any variations $\delta \alpha_i$ and $\delta \beta_j^*$ we have the **discrete Euler-Lagrange Equations**:

$$\frac{\delta S}{\delta \alpha_i} + d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) = 0 \quad (4.1)$$

$$\frac{\delta S}{\delta \beta_j^*} + (d_{n-k_j-1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) = 0 \quad (4.2)$$

Unlike in smooth theories where the Euler-Lagrange equations are written in terms of the Lagrangian, our discrete Euler-Lagrange equations are written in terms of the action. This result mirrors a similar result in discrete mechanics. In [Marsden and West 2001] page 363 the discrete Lagrangian L_d is in fact a discrete action between adjacent points. This may not be a surprise since in the discrete setting quantities are naturally integrated and not defined point-wise.

Example 4.1.4. Recall that complex scalar field theory has an action given by:

$$S = \frac{1}{2} (d^0 \phi, d^0 \bar{\phi}) + \frac{m}{2} (\phi, \bar{\phi}) .$$

The Euler-Lagrange equations for this field theory are then given by:

$$\begin{aligned} \frac{\delta S}{\delta \phi} + d_1^* \left(\frac{\delta S}{\delta d^0 \phi} \right) &= 0 \\ \frac{\delta S}{\delta \bar{\phi}} + d_1^* \left(\frac{\delta S}{\delta d^0 \bar{\phi}} \right) &= 0 \end{aligned}$$

Since $\frac{\delta S}{\delta \phi} = \frac{m}{2} \bar{\phi}$ as was shown above and $\frac{\delta S}{\delta d^0 \phi} = \frac{1}{2} d^0 \bar{\phi}$ through a similar calculation we derive:

$$\begin{aligned} m \bar{\phi} + d_1^* d^0 \bar{\phi} &= 0 \\ m \phi + d_1^* d^0 \phi &= 0 , \end{aligned}$$

which matches the standard result from variational mechanics, which is often written as:

$$\begin{aligned} m\bar{\phi} + \Delta_0\bar{\phi} &= 0 \\ m\phi + \Delta_0\phi &= 0 , \end{aligned}$$

where $\Delta_0 = d_1^* d^0$ is the discrete Laplacian on zero-cochains.

Example 4.1.5. Real $O(n)$ field theory is again similar to the complex scalar field theory case. The action is given by:

$$S = \frac{1}{2} (d^0 \phi, d^0 \phi) + \frac{m}{2} (\phi, \phi) .$$

Recall that the discrete exterior derivative acts component by component on the vector dimensions of ϕ and that the inner product here is given by thinking of transposing both the vector space and cochain dimensions as described in Example 4.1.2. The Euler-Lagrange equations for this field theory are given by:

$$m\phi - d_1^* d^0 \phi = 0 .$$

Example 4.1.6. Abelian Yang-Mills is given by the action:

$$S = \frac{1}{2} (d^1 A, d^1 A) .$$

The Euler-Lagrange equations for this field theory are given by:

$$d_2^* d^1 A = 0 .$$

4.1.2. Noether's First Theorem

In the smooth setting, Noether's first theorem proves that corresponding to any global continuous symmetry there is a conserved current. We will derive an analogous quantity in the discrete setting, where again the adjointness of d and d^* is the key mathematical tool needed. Unlike the above discussion, this will be derived for primal 0-cochains and dual 0-cochains. This covers most of the physically-relevant systems, and we discuss the reasoning for this restriction in Remark 4.1.9. In the smooth setting, Noether's theorem is derived by looking for variations that change the action by a total derivative. Integrating the Lagrangian, this means that the action can change by a boundary term. While Noether's theorem is proved only for 0-cochains the following lemma is proved for cochains of any degree.

Lemma 4.1.7. *Given primal k_i -cochains α_i and dual $(n - k_j)$ -cochains β_j^* with a symmetry variation, written as $\delta\alpha_i = X_i(\alpha_i)$ and $\delta\beta_j^* = Y_j(\beta_j^*)$, respectively, where X_i are functions from primal cochains to primal cochains and Y_j are functions from dual cochains to dual cochains. A*

symmetry transformation is one such that the variation of the action is entirely on the boundary (see Remark 4.1.8):

$$\begin{aligned}\delta S &= \sum_i \sum_{\star \sigma^{k_i} \cap \partial K} \langle H_i^{*\partial}, \star \sigma^{k_i} \cap \partial K \rangle + \sum_j \sum_{\sigma^{k_j-1} \in \partial K} \langle H_j, \sigma^{k_j-1} \rangle \\ &= \sum_i \tilde{1}^T i_{k_i}^\partial H_i^{*\partial} + \sum_j \left(i_{k_j-1}^\partial \tilde{1} \right)^T H_j ,\end{aligned}$$

where $H_i^{*\partial}$ are boundary dual $(n-1-k_i)$ -cochains and H_j are primal (k_j-1) -cochains. Then we have the equality

$$\begin{aligned}0 &= \sum_i \left[(X_i \alpha_i)^T i_{k_i}^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] - \tilde{1}^T i_{k_i}^\partial H_i^{*\partial} \right] \\ &\quad + \sum_j \left[(-1)^{k_j-1} \left(i_{k_j-1}^\partial (Y_j \beta_j^*)^\partial \right)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) - \left(i_{k_j-1}^\partial \tilde{1} \right)^T H_j \right] .\end{aligned}$$

Proof. A variation of the action can be written:

$$\begin{aligned}\delta S &= \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \right] \\ &\quad + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) \right] .\end{aligned}$$

Inserting the Euler-Lagrange Equations 4.1,4.2 for $\frac{\delta S}{\delta \alpha_i}$ and $\frac{\delta S}{\delta \beta_j^*}$, respectively gives:

$$\begin{aligned}\delta S &= \sum_i \left[- \left(d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \right] \\ &\quad \sum_j \left[- \left((d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) \right] .\end{aligned}$$

This can be further simplified by inserting the form of the variations of α_i and β_j^* :

$$\begin{aligned}\delta S &= \sum_i \left[- \left(d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), X_i(\alpha_i) \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} X_i(\alpha_i) \right) \right] \\ &\quad \sum_j \left[- \left((d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), Y_j(\beta_j^*) \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} Y_j(\beta_j^*) \right) \right] .\end{aligned}$$

Using adjointness of d and d^* , Propositions 3.2.3 and 3.2.5, we obtain only boundary terms:

$$\begin{aligned} \delta S = & \sum_i \left[(X_i(\alpha_i))^T i_{k_i}^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] \right] \\ & + \sum_j \left[(-1)^{k_j-1} \left(i_{k_j-1}^\partial (Y_j(\beta_j^*))^\partial \right)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) \right]. \end{aligned}$$

equating this with the form of δS from the assumptions we have:

$$\begin{aligned} 0 = & \sum_i \left[(X_i(\alpha_i))^T i_{k_i}^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] - \vec{1}^T i_{k_i}^\partial H_i^* \right] \\ & + \sum_j \left[(-1)^{k_j-1} \left(i_{k_j-1}^\partial (Y_j(\beta_j^*))^\partial \right)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) - \left(i_{k_j-1}^\partial \vec{1} \right)^T H_j \right]. \end{aligned}$$

□

Remark 4.1.8. We allow boundary terms because in the smooth setting symmetry transformations are the ones that change the Lagrangian by a total derivative, which when integrated give a boundary term.

Remark 4.1.9. To prove Noether's theorem we will need to relate the result of the above lemma to a discrete integral of a total derivative, which we will do with Proposition 3.3.4 which was only proven for the pairs of a primal 0-cochain and dual $(n-1)$ cochain as well as a primal $(n-1)$ -cochain and dual 0-cochain. Generalizing that proposition to general cochains would allow for a more general Noether's theorem.

Theorem 4.1.10 (Fully-discrete Noether's theorem). *Let ϕ_i denote a collection of primal 0-cochains, ϕ_j^* denote a collection dual 0-cochains, H_i^* denote a collection of dual $(n-1)$ -cochains and H_j denote a collection of primal $(n-1)$ -cochains. Given the assumptions of the above lemma we have a primal Noether current and dual Noether current:*

$$\begin{aligned} J &:= \sum_j (-1)^{n-1} Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} - H_j \\ J^* &:= \sum_i X_i(\phi_i) \wedge *_1 \frac{\delta S}{\delta d^0 \phi_i} - H_i^*, \end{aligned}$$

such that:

$$\sum_{\sigma^0} \langle d_{n-1}^{\text{dual}} J^*, \star \sigma^0 \rangle + \sum_{\sigma^n} \langle d^{n-1} J, \sigma^n \rangle = 0$$

Proof. From Lemma 4.1.7 above we have the equality:

$$0 = \sum_i \left[(X_i(\phi_i))^T i_0^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] - \vec{1}^T i_0^\partial H_i^{*\partial} \right] \\ + \sum_j \left[(-1)^{n-1} \left(i_{n-1}^\partial (Y_j(\phi_j^*))^\partial \right)^T *_{n-1}^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - (i_{n-1}^\partial \vec{1})^T H_j \right].$$

And from Proposition 3.3.4 we have:

$$0 = \sum_{\sigma^0} \sum_i \langle d_{n-1}^{\text{dual}} \left(X_i(\phi_i) \wedge *_{-1} \left(\frac{\delta S}{\delta d^0 \phi_i} \right) - \vec{1} \wedge H_i^* \right), \star \sigma^0 \rangle \\ + \sum_{\sigma^n} \sum_j (-1)^{n-1} \langle d^{n-1} \left(Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - H_j \wedge \vec{1} \right), \sigma^n \rangle \\ = \sum_{\sigma^0} \sum_i \langle d_{n-1}^{\text{dual}} \left(X_i(\phi_i) \wedge *_{-1} \left(\frac{\delta S}{\delta d^0 \phi_i} \right) - H_i^* \right), \star \sigma^0 \rangle \\ + \sum_{\sigma^n} \sum_j (-1)^{n-1} \langle d^{n-1} \left(Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - H_j \right), \sigma^n \rangle,$$

where in the last step we used that for a dual $(n-1)$ -cochain H_i^* and 0-cochain $\vec{1}$ that $\vec{1} \wedge H_i^* = H_i^*$ and likewise for a primal $(n-1)$ -cochain H_j and dual 0-cochain $\vec{1}$ that $H_j \wedge \vec{1} = H_j$. \square

Example 4.1.11. The symmetry for complex scalar field theory is the multiplication of ϕ by $e^{i\alpha}$ where α is a real number. Under this transformation our fields transform as:

$$\phi \mapsto \phi' = e^{i\alpha} \phi \\ \bar{\phi} \mapsto \bar{\phi}' = e^{-i\alpha} \bar{\phi},$$

and since $d^0 e^{i\alpha} \phi = e^{i\alpha} d^0 \phi$, the action remains unchanged:

$$S \mapsto S' = \frac{1}{2} (d^0 e^{i\alpha} \phi, d^0 e^{-i\alpha} \bar{\phi}) + \frac{m}{2} (e^{i\alpha} \phi, e^{-i\alpha} \bar{\phi}) \\ = \frac{1}{2} (e^{i\alpha} d^0 \phi, e^{-i\alpha} d^0 \bar{\phi}) + \frac{m}{2} (e^{i\alpha} \phi, e^{-i\alpha} \bar{\phi}) \\ = \frac{1}{2} (d^0 \phi, d^0 \bar{\phi}) + \frac{m}{2} (\phi, \bar{\phi}),$$

where the last step follows from the definition of the real inner product used above. The conserved current is therefore:

$$J^* = i\phi \wedge *_{-1} \frac{\delta S}{\delta d^0 \phi} - i\bar{\phi} \wedge *_{-1} \frac{\delta S}{\delta d^0 \bar{\phi}} \\ = i\phi \wedge *_{-1} \frac{1}{2} d^0 \bar{\phi} - i\bar{\phi} \wedge *_{-1} \frac{1}{2} d^0 \phi$$

$$= \frac{i}{2} \left[\phi \wedge *_1 d^0 \bar{\phi} - \bar{\phi} \wedge *_1 d^0 \phi \right] .$$

Since the action is invariant under the gauge transformation the H_i^* from Theorem 4.1.10 are zero.

Example 4.1.12. The symmetry for $O(n)$ field theory is similar to the one for scalar field theory. The symmetry transformation is multiplication of ϕ by R where R is a $n \times n$ real rotation matrix. We can always write that matrix as $R = e^r$, where r is the generator of the rotation. The field then transforms as:

$$\phi \mapsto \phi' = R\phi$$

and since $d^0 R\phi = R d^0 \phi$, the action remains unchanged:

$$\begin{aligned} S \mapsto S' &= \frac{1}{2} (d^0 R\phi, d^0 R\phi) + \frac{m}{2} (R\phi, R\phi) \\ &= \frac{1}{2} (R d^0 \phi, R d^0 \phi) + \frac{m}{2} (R\phi, R\phi) \\ &= \frac{1}{2} (d^0 \phi, d^0 \phi) + \frac{m}{2} (\phi, \phi) , \end{aligned}$$

where the last step follows from the definition of the inner product. The conserved current is now:

$$\begin{aligned} J^* &= i\phi \wedge *_1 \frac{\delta S}{\delta d^0 \phi} \\ &= \phi \wedge *_1 d^0 \phi \\ &= \phi \wedge *_1 d^0 \phi \end{aligned}$$

Remark 4.1.13. We do not show the currents for discrete abelian Yang-Mills because the fields are 1-cochains and Theorem 4.1.10 was only proved for theories involving 0-cochains.

4.2. Semi-Discrete Field Theories

We will re-derive the results of the preceding section for semi-discrete field theories, that is theories where time is left continuous and space is discretized. The proofs for semi-discrete field theories are similar to those for the fully-discrete theories. However, the principal difference is that the inner product is only over the spatial dimensions and that the time components of the discretized differential forms needs to be explicitly accounted for separately from the spatial components adding to the length of the expressions and proofs. However, the general strategies remain the same as in the fully-discrete case. Again we will illustrate our work with physical examples, namely complex scalar field theory, $O(n)$ field theory, and abelian Yang-Mills (electrodynamics).

One complication with semi-discrete theories is that since the temporal component of a smooth differential form is not discretized the discretized form will become a cochain defined in two different

dimensions. For example, consider the two dimensional cochain

$$\alpha = \alpha_1 dt \wedge dx + \alpha_2 dx \wedge dy ,$$

in three-dimensional space time. $\alpha_2 dx \wedge dy$ will be discretized by integrating the form on the two dimensional cells and $\alpha_1 dt \wedge dx$ will be discretized by integrating the form on the one dimensional cells. In our discrete models we will use the superscript t to denote the temporal components which are integrated on the one-dimension lower cells. Our discrete action will involve primal cochains α_i each of spatial degree k_i (except the temporal components α_i^t of spatial degree $(k_i - 1)$) and dual cochains β_j^* each of spatial degree $(n - k_j)$ (and again except the temporal components β_j^{t*} of spatial dimension $(n - k_j - 1)$). The action, S , can be constructed from linear combinations of inner products of α_i , α_i^t , β_j^* , β_j^{t*} , $d^{k_i} \alpha_i$, $d^{k_i-1} \alpha_i^t$, $\partial_t \alpha_i$, $d_{n-k_j}^{\text{dual}} \beta_j^*$, $\partial_t \beta_j^*$, $d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}$ along with the necessary hodge stars. Note that there are no terms like $\partial_t \alpha_i^t$ or $\partial_t \beta_j^{t*}$ since $dt \wedge dt = 0$. As in the fully-discrete case, most of the possible terms do not appear in physically-relevant examples, and we will illustrate our general proofs using the same examples as before: complex scalar field theory, $O(n)$ field theory and abelian Yang-Mills.

Example 4.2.1. Complex scalar field theory is defined by the action:

$$S = \int \frac{1}{2} [(\partial_t \phi, \partial_t \bar{\phi}) + (d^0 \phi, d^0 \bar{\phi})] + \frac{m}{2} (\phi, \bar{\phi}) dt ,$$

where ϕ is complex-valued primal zero-cochain and $\bar{\phi}$ is its complex-conjugate. As in the fully-discrete case, since ϕ is complex it is actually made of two independent, real components $\phi = \phi_1 + i\phi_2$, which we will need to later allow to vary independently. However, it is more common to instead think of ϕ and $\bar{\phi}$ as being the independent variables. This amounts to a change of basis from the $\{\phi_1, \phi_2\}$ basis to the $\{\phi, \bar{\phi}\}$.

Example 4.2.2. The $O(n)$ field theory is similar to the complex scalar field theories with a real n -dimensional vector replacing the complex scalar field on each zero-cell. The action is defined by:

$$S = \int \frac{1}{2} [(\partial_t \phi, \partial_t \phi) + (d^0 \phi, d^0 \phi)] + \frac{m}{2} (\phi, \phi) dt ,$$

where ϕ are real n -dimensional vectors on each zero-cell. Here d^0 is the usual d^0 applied to every entry in the vector and the inner product is formed by taking the transpose of the first term both in the vector dimensions and in the cochain dimensions. Also note that each ϕ has in this case n independent components that will need to be varied independently, as in the complex scalar case. In fact, the complex scalar theory is isomorphic to the $O(2)$ theory.

Example 4.2.3. Abelian Yang-Mills theory is defined by the action:

$$S = \int \frac{1}{2} (\partial_t A, \partial_t A) + (d^0 A^t, \partial_t A) + \frac{1}{2} (d^0 A^t, d^0 A^t) + \frac{1}{2} (d^1 A, d^1 A) dt ,$$

where A is a real one-cochain and A^t is a real zero-cochain. For a discussion of Yang-Mills more generally see Chapter 5.

4.2.1. Euler-Lagrange Equations

To derive the Euler-Lagrange equations we will follow the same strategy as for the fully-discrete theory though there will be more terms and an extra step due to how the time component is treated separately. First we will vary the fields which will induce variations on the derivatives and a variation of the action. We then can apply adjointness of d and d^* as well as integration by parts in time. Finally we can regroup and find our result. However first, need to vary the action with respect to the perturbations of the fields:

$$\begin{aligned}\alpha_i &\mapsto \alpha_i + \delta\alpha_i & \alpha_i^t &\mapsto \alpha_i^t + \delta\alpha_i^t \\ \beta_j^* &\mapsto \beta_j^* + \delta\beta_j^* & \beta_j^{t*} &\mapsto \beta_j^{t*} + \delta\beta_j^{t*},\end{aligned}$$

which induces the variations on the derivatives:

$$\begin{aligned}d^{k_i} \alpha_i &\mapsto d^{k_i} \alpha_i + d^{k_i} \delta\alpha_i \\ d^{k_i-1} \alpha_i^t &\mapsto d^{k_i-1} \alpha_i^t + d^{k_i-1} \delta\alpha_i^t \\ \partial_t \alpha_i &\mapsto \partial_t \alpha_i + \partial_t \delta\alpha_i \\ d_{n-k_j}^{\text{dual}} \beta_j^* &\mapsto d_{n-k_j}^{\text{dual}} \beta_j^* + d_{n-k_j}^{\text{dual}} \delta\beta_j^* \\ d_{n-k_j-1}^{\text{dual}} \beta_j^{t*} &\mapsto d_{n-k_j-1}^{\text{dual}} \beta_j^{t*} + d_{n-k_j-1}^{\text{dual}} \delta\beta_j^{t*} \\ \partial_t \beta_j^* &\mapsto \partial_t \beta_j^* + \partial_t \delta\beta_j^*.\end{aligned}$$

This imposes variations on S :

$$\begin{aligned}\delta S = \int \Bigg\{ \sum_i \Bigg[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \\ + \left(\frac{\delta S}{\delta \alpha_i^t}, \delta \alpha_i^t \right) + \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, d^{k_i-1} \delta \alpha_i^t \right) \Bigg] \\ + \sum_j \Bigg[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) \\ + \left(\frac{\delta S}{\delta \beta_j^{t*}}, \delta \beta_j^{t*} \right) + \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}}, d_{n-k_j-1}^{\text{dual}} \delta \beta_j^{t*} \right) \Bigg] \Bigg\} dt.\end{aligned}$$

We can now apply adjointness of d and d^* proved in Proposition 3.2.3 and 3.2.5. As usual for derivations of the Euler-Lagrange equations, we will not allow our fields to vary on the boundary

and so no boundary terms will appear.

$$\begin{aligned} \delta S = \int \left\{ \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t \delta \alpha_i \right) + \left(d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \alpha_i^t}, \delta \alpha_i^t \right) + \left(d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right), \delta \alpha_i^t \right) \right] \right. \\ \left. + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t \delta \beta_j^* \right) + \left((d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \beta_j^{t*}}, \delta \beta_j^{t*} \right) + \left((d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), \delta \beta_j^{t*} \right) \right] \right\} dt. \end{aligned}$$

Integrating by parts over time gives:

$$\begin{aligned} \delta S = \int \left\{ \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) - \left(\partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right), \delta \alpha_i \right) + \left(d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \alpha_i^t}, \delta \alpha_i^t \right) + \left(d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right), \delta \alpha_i^t \right) \right] \right. \\ \left. + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) - \left(\partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right), \delta \beta_j^* \right) + \left((d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \beta_j^{t*}}, \delta \beta_j^{t*} \right) + \left((d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), \delta \beta_j^{t*} \right) \right] \right\} dt. \end{aligned}$$

Combining the terms gives:

$$\begin{aligned} \delta S = \int \left\{ \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i} - \partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right) + d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \alpha_i^t} + d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right), \delta \alpha_i^t \right) \right] \right. \\ \left. + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*} - \partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right) + (d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) \right. \right. \\ \left. \left. + \left(\frac{\delta S}{\delta \beta_j^{t*}} + (d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), \delta \beta_j^{t*} \right) \right] \right\} dt. \end{aligned}$$

Since this must be zero for any variations $\delta \alpha_i$, $\delta \alpha_i^t$, $\delta \beta_j^*$, and $\delta \beta_j^{t*}$ we have the **semi-discrete Euler-Lagrange Equations**:

$$\frac{\delta S}{\delta \alpha_i} - \partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right) + d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) = 0 \quad (4.3)$$

$$\frac{\delta S}{\delta \alpha_i^t} + d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right) = 0 \quad (4.4)$$

$$\frac{\delta S}{\delta \beta_j^*} - \partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right) + (d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) = 0 \quad (4.5)$$

$$\frac{\delta S}{\delta \beta_j^{t*}} + (d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right) = 0 \quad (4.6)$$

Example 4.2.4. Recall that complex scalar field theory has an action given by:

$$S = \int \frac{1}{2} [(\partial_t \phi, \partial_t \bar{\phi}) + (d^0 \phi, d^0 \bar{\phi})] + \frac{m}{2} (\phi, \bar{\phi}) dt .$$

The Euler-Lagrange equations for this field theory are then given by:

$$\begin{aligned} \frac{\delta S}{\delta \phi} - \partial_t \left(\frac{\delta S}{\delta \partial_t \phi} \right) + d_1^* \left(\frac{\delta S}{\delta d^0 \phi} \right) &= 0 \\ \frac{\delta S}{\delta \bar{\phi}} - \partial_t \left(\frac{\delta S}{\delta \partial_t \bar{\phi}} \right) + d_1^* \left(\frac{\delta S}{\delta d^0 \bar{\phi}} \right) &= 0 . \end{aligned}$$

Since

$$\frac{\delta S}{\delta \phi} = \frac{m}{2} \bar{\phi} \quad \text{and} \quad \frac{\delta S}{\delta d^0 \phi} = \frac{1}{2} d^0 \bar{\phi} ,$$

through a similar calculation to what was done for the fully-discrete theory and

$$\frac{\delta S}{\delta \partial_t \phi} = \frac{1}{2} \partial_t \bar{\phi} ,$$

we derive:

$$\begin{aligned} m \bar{\phi} - \partial_t^2 \bar{\phi} + d_1^* d^0 \bar{\phi} &= 0 \\ m \phi - \partial_t^2 \phi + d_1^* d^0 \phi &= 0 . \end{aligned}$$

This matches the standard result from variational mechanics, as was the case in the fully-discrete theory. Recall that $\Delta_0 = d_1^* d^0$ is the discrete Laplacian on zero-cochains.

Example 4.2.5. Real $O(n)$ field theory is again similar to the complex scalar field theory case. The action is given by:

$$S = \int \frac{1}{2} [(\partial_t \phi, \partial_t \phi) + (d^0 \phi, d^0 \phi)] + \frac{m}{2} (\phi, \phi) dt .$$

Recall that the discrete exterior derivative acts component by component on the vector dimensions of ϕ and that the inner product here is given by thinking of transposing both the vector space

and cochain dimensions as described in Example 4.1.2. The Euler-Lagrange equations for this field theory are given by:

$$m\phi - \partial_t^2 \phi + d_1^* d^0 \phi = 0 .$$

Example 4.2.6. Abelian Yang-Mills is given by the action:

$$S = \int \frac{1}{2} (\partial_t A, \partial_t A) + (d^0 A^t, \partial_t A) + \frac{1}{2} (d^0 A^t, d^0 A^t) + \frac{1}{2} (d^1 A, d^1 A) dt ,$$

The Euler-Lagrange equations for this field theory are given by:

$$\begin{aligned} -\partial_t^2 A - \partial_t d^0 A^t + d_2^* d^1 A &= 0 \\ d_1^* \partial_t A + d_1^* d^0 A^t &= 0 . \end{aligned}$$

Interpreting A^t as the electric potential, V and A as the magnetic vector potential A we can identify the electric and magnetic fields:

$$\begin{aligned} B &= d^1 A \\ E &= -d^0 A^t - \partial_t A , \end{aligned}$$

we have the discrete charge-less Maxwell equations:

$$\begin{aligned} \partial_t E + d_2^* B &= 0 \\ d^1 E &= 0 . \end{aligned}$$

4.2.2. Noether's First Theorem

In the smooth setting, Noether's first theorem proves that to any global continuous symmetry there is a conserved current. Unlike in the fully-discrete models where both space and time are discretized together, since time is left separate in semi-discrete models, Noether's theorem is easier to interpret as a charge density and a current density. As in the fully-discrete case, we derive Noether's theorem for primal zero-cochains and dual zero-cochains, which we denote ϕ_i and ϕ_j^* , though the following lemma is for any degree cochains.

Lemma 4.2.7. *Given primal k_i -cochains α_i , primal $(k_i - 1)$ -cochains α_i^t , dual $(n - k_j)$ -cochains β_j^* and dual $(n - k_j - 1)$ -cochains β_j^{*t} with a symmetry variation, written as $\delta\alpha_i = X_i(\alpha_i)$, $\delta\alpha_i^t = X_i^t(\alpha_i^t)$, $\delta\beta_j^* = Y_j(\beta_j^*)$, and $\delta\beta_j^{*t} = Y_j^t(\beta_j^{*t})$, respectively, where X_i are functions from primal cochains to primal cochains and Y_j are functions from dual cochains to dual cochains. A symmetry transformation is one such that the variation of the action is entirely on the boundary*

(see Remark 4.1.8):

$$\begin{aligned}
\delta S &= \sum_i \left[\sum_{\star \sigma^{k_i} \cap \partial K} \int \langle H_i^{*\partial}, \star \sigma^{k_i} \cap \partial K \rangle dt + \sum_{\sigma^{k_i}} \langle \tilde{H}_i^*, \star \sigma^{k_i} \rangle \Big|_{t_i}^{t_f} \right. \\
&\quad \left. + \sum_{\star \sigma^{k_i+1} \cap \partial K} \int \langle H_i^{*t\partial}, \star \sigma^{k_i+1} \cap \partial K \rangle dt \right] \\
&+ \sum_j \left[\sum_{\sigma^{k_j} \in \partial K} \langle H_j, \sigma^{k_j} \rangle + \sum_{\sigma^{k_j-1} \in \partial K} \int \langle H_j^t, \sigma^{k_j-1} \rangle dt + \sum_{\sigma^{k_j}} \langle \tilde{H}_j, \sigma^{k_j} \rangle \Big|_{t_i}^{t_f} \right] \\
&= \sum_i \left[\int \bar{1}^T i_{k_i}^\partial H_i^{*\partial} + \bar{1}^T i_{k_i-1}^\partial H_i^{*t\partial} dt + (\tilde{H}_i^*, *_{k_i} \bar{1}) \Big|_{t_i}^{t_f} \right] \\
&+ \sum_j \left[\int (i_{k_j-1}^\partial \bar{1})^T H_j + (i_{k_j}^\partial \bar{1})^T H_j^t dt + (\tilde{H}_j, *_{k_j}^{-1} \bar{1}) \Big|_{t_i}^{t_f} \right].
\end{aligned}$$

where $H_i^{*\partial}$ are boundary dual $(n-1-k_i)$ -cochains, $H_i^{*t\partial}$ are boundary dual $(n-k_i)$ -cochains, \tilde{H}_i^* are internal dual $(n-k_i)$ -cochains, H_j^t are primal k_j -cochains, H_j are primal (k_j-1) -cochains, \tilde{H}_j are primal k_j -cochains, and t_i and t_j are the initial and final times. We then have the equality:

$$\begin{aligned}
0 &= \sum_i \left[\int \left((X_i(\alpha_i))^T i_{k_i}^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] - \bar{1}^T i_{k_i}^\partial H_i^{*\partial} \right. \right. \\
&\quad \left. \left. + (X_i^t(\alpha_i^t))^T i_{k_i-1}^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right) \right] - \bar{1}^T i_{k_i-1}^\partial H_i^{*t\partial} \right) dt \right. \\
&\quad \left. + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, X_i(\alpha_i) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_i^*, *_{k_i} \bar{1}) \Big|_{t_i}^{t_f} \right] \\
&+ \sum_j \left[\int \left((-1)^{k_j-1} (i_{k_j-1}^\partial (Y_j(\beta_j^*))^\partial)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) - (i_{k_j-1}^\partial \bar{1})^T H_j \right. \right. \\
&\quad \left. \left. + (-1)^{k_j} (i_{k_j}^\partial (Y_j^t(\beta_j^{t*}))^\partial)^T *_{k_j}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right) - (i_{k_j}^\partial \bar{1})^T H_j^t \right) dt \right. \\
&\quad \left. + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_j, *_{k_j}^{-1} \bar{1}) \Big|_{t_i}^{t_f} \right].
\end{aligned}$$

Proof. A variation of the action can be written:

$$\begin{aligned}
\delta S &= \int \left\{ \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta \alpha_i \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t \delta \alpha_i \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) \right. \right. \\
&\quad \left. \left. + \left(\frac{\delta S}{\delta \alpha_i^t}, \delta \alpha_i^t \right) + \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, d^{k_i-1} \delta \alpha_i^t \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) \right. \\
& \left. + \left(\frac{\delta S}{\delta \beta_j^{t*}}, \delta \beta_j^{t*} \right) + \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}}, d_{n-k_j-1}^{\text{dual}} \delta \beta_j^{t*} \right) \right] dt.
\end{aligned}$$

Inserting the Euler-Lagrange Equations (4.3), (4.4), (4.5), and (4.6) for $\frac{\delta S}{\delta \alpha_i}$, $\frac{\delta S}{\delta \alpha_i^t}$, $\frac{\delta S}{\delta \beta_j^*}$ and $\frac{\delta S}{\delta \beta_j^{t*}}$, respectively gives:

$$\begin{aligned}
\delta S = \int \left\{ \sum_i \left[\left(\partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right) - d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), \delta \alpha_i \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t \delta \alpha_i \right) \right. \right. \\
+ \left. \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} \delta \alpha_i \right) - d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, \delta \alpha_i^t \right) + \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, d^{k_i-1} \delta \alpha_i^t \right) \right] \\
+ \sum_j \left[\left(\partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right) - (d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), \delta \beta_j^* \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t \delta \beta_j^* \right) \right. \\
+ \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} \delta \beta_j^* \right) - \left((d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), \delta \beta_j^{t*} \right) \\
\left. \left. + \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}}, d_{n-k_j-1}^{\text{dual}} \delta \beta_j^{t*} \right) \right] \right\} dt.
\end{aligned}$$

This can be further simplified by inserting the form of the variations of α_i and β_j^* and by saying hi to the reader:

$$\begin{aligned}
\delta S = \int \left\{ \sum_i \left[\left(\partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right) - d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), X_i(\alpha_i) \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t X_i(\alpha_i) \right) \right. \right. \\
+ \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} X_i(\alpha_i) \right) - d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, X_i^t(\alpha_i^t) \right) \\
\left. \left. + \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, d^{k_i-1} X_i^t(\alpha_i^t) \right) \right] \right. \\
+ \sum_j \left[\left(\partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right) - (d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), Y_j(\beta_j^*) \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t Y_j(\beta_j^*) \right) \right. \\
+ \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} Y_j(\beta_j^*) \right) - \left((d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), Y_j^t(\beta_j^{t*}) \right) \\
\left. \left. + \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}}, d_{n-k_j-1}^{\text{dual}} Y_j^t(\beta_j^{t*}) \right) \right] \right\} dt \\
= \int \left\{ \sum_i \left[\left(-d_{k_i+1}^* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right), X_i(\alpha_i) \right) + \left(\frac{\delta S}{\delta d^{k_i} \alpha_i}, d^{k_i} X_i(\alpha_i) \right) \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -d_{k_i}^* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, X_i^t(\alpha_i^t) \right) + \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t}, d^{k_i-1} X_i^t(\alpha_i^t) \right) \\
& + \left(\partial_t \left(\frac{\delta S}{\delta \partial_t \alpha_i} \right), X_i(\alpha_i) \right) + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, \partial_t X_i(\alpha_i) \right) \Big] \\
& + \sum_j \left[- \left((d_{n-k_j+1}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right), Y_j(\beta_j^*) \right) + \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*}, d_{n-k_j}^{\text{dual}} Y_j(\beta_j^*) \right) \right. \\
& - \left((d_{n-k_j}^{\text{dual}})^* \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right), Y_j^t(\beta_j^{t*}) \right) + \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}}, d_{n-k_j-1}^{\text{dual}} Y_j^t(\beta_j^{t*}) \right) \\
& \left. + \left(\partial_t \left(\frac{\delta S}{\delta \partial_t \beta_j^*} \right), Y_j(\beta_j^*) \right) + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, \partial_t Y_j(\beta_j^*) \right) \right] \Big\} dt.
\end{aligned}$$

Using adjointness of d and d^* , Proposition 3.2.3 and 3.2.5, as well as integrating by parts in time we obtain only boundary terms:

$$\begin{aligned}
\delta S = & \sum_i \left[\int \left((X_i(\alpha_i))^T i_{k_i}^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] \right. \right. \\
& + (X_i^t(\alpha_i^t))^T i_{k_i-1}^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right) \right] \Big) dt \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, X_i(\alpha_i) \right) \Big|_{t_i}^{t_f} \right] \\
& + \sum_j \left[\int (-1)^{k_j-1} \left((i_{k_j-1}^\partial (Y_j(\beta_j^*)))^\partial \right)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) \right. \\
& + (-1)^{k_j} \left(i_{k_j}^\partial (Y_j^t(\beta_j^{t*}))^\partial \right)^T *_{k_j}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right) \Big) dt \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} \right].
\end{aligned}$$

Matching terms with the form of the variation of the action we have:

$$\begin{aligned}
0 = & \sum_i \left[\int \left((X_i(\alpha_i))^T i_{k_i}^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i} \alpha_i} \right) \right] - \vec{1}^T i_{k_i}^\partial H_i^{*\partial} \right. \right. \\
& + (X_i^t(\alpha_i^t))^T i_{k_i-1}^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^{k_i-1} \alpha_i^t} \right) \right] - \vec{1}^T i_{k_i-1}^\partial H_i^{*t\partial} \Big) dt \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \alpha_i}, X_i(\alpha_i) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_i^*, *_{k_i} \vec{1}) \Big|_{t_i}^{t_f} \right] \\
& + \sum_j \left[\int \left((-1)^{k_j-1} (i_{k_j-1}^\partial (Y_j(\beta_j^*)))^\partial \right)^T *_{k_j-1}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j}^{\text{dual}} \beta_j^*} \right) - (i_{k_j-1}^\partial \vec{1})^T H_j \right. \\
& \left. + (-1)^{k_j} (i_{k_j}^\partial (Y_j^t(\beta_j^{t*}))^\partial)^T *_{k_j}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right) - (i_{k_j}^\partial \vec{1})^T H_j^t \right] dt
\end{aligned}$$

$$\begin{aligned}
& +(-1)^{k_j} \left(i_{k_j}^\partial \left(Y_j^t(\beta_j^{t*}) \right)^\partial \right)^T *_{k_j}^{-1} \left(\frac{\delta S}{\delta d_{n-k_j-1}^{\text{dual}} \beta_j^{t*}} \right) - \left(i_{k_j}^\partial \vec{1} \right)^T H_j^t \Big) dt \\
& + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - \left(\tilde{H}_j, *_{k_j}^{-1} \vec{1} \right) \Big|_{t_i}^{t_f} \Big].
\end{aligned}$$

Where $\left(\frac{\delta S}{\delta \partial_t \alpha_i}, X_i(\alpha_i) \right) \Big|_{t_i}^{t_f} - \left(\tilde{H}_i^*, *_{k_i} \vec{1} \right) \Big|_{t_i}^{t_f}$ should be interpreted as:

$$\begin{aligned}
& \left(\frac{\delta S}{\delta \partial_t \alpha_i}, X_i(\alpha_i) \right) \Big|_{t_i}^{t_f} - \left(\tilde{H}_i^*, *_{k_i} \vec{1} \right) \Big|_{t_i}^{t_f} \\
& = \sum_{\sigma^{k_i}} (-1)^{k_i(n-k_i)} \langle X_i(\alpha_i) \wedge *_{k_i} \frac{\delta S}{\delta \partial_t \alpha_i} - \vec{1} \wedge \tilde{H}_i, V_{\sigma^{k_i}} \rangle \Big|_{t_i}^{t_f}.
\end{aligned}$$

And likewise $\left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - \left(\tilde{H}_j, *_{k_j}^{-1} \vec{1} \right) \Big|_{t_i}^{t_f}$ should be interpreted as:

$$\left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - \left(\tilde{H}_j, *_{k_j}^{-1} \vec{1} \right) \Big|_{t_i}^{t_f} = \sum_{\sigma^{k_j}} \langle Y_j(\beta_j^*) \wedge \frac{\delta S}{\delta \partial_t \beta_j^*} - \vec{1} \wedge \tilde{H}_j, V_{\sigma^{k_j}} \rangle \Big|_{t_i}^{t_f}.$$

□

Theorem 4.2.8 (Semi-discrete Noether's theorem). *Let ϕ_i denote a collection of primal 0-cochains, ϕ_j^* denote a collection dual 0-cochains, H_i^* denote a collection of dual $(n-1)$ -cochains and H_j denote a collection of primal $(n-1)$ -cochains. Given the assumptions of the above lemma we have a primal Noether current and dual Noether current:*

$$\begin{aligned}
J &:= \sum_j (-1)^{n-1} Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} - H_j \\
J^* &:= \sum_i Y_i(\phi_i) \wedge *_1 \frac{\delta S}{\delta d^0 \phi_i} - H_i^*,
\end{aligned}$$

and primal and dual Noether charge:

$$\begin{aligned}
Q &:= \sum_j Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \frac{\delta S}{\delta \partial_t \phi_j^*} - \tilde{H}_j \\
Q^* &:= \sum_i X_i(\phi_i) \wedge *_0 \frac{\delta S}{\delta \partial_t \phi_i} - \tilde{H}_i^*
\end{aligned}$$

such that:

$$\int \sum_{\sigma^n} \langle \partial_t Q + d^{n-1} J, \sigma^n \rangle + \sum_{\sigma^0} \langle \partial_t Q^* + d_{n-1}^{\text{dual}} J^*, \star \sigma^0 \rangle dt = 0$$

Proof. From Lemma 4.2.7 above we have the equality:

$$\begin{aligned}
0 = & \sum_i \left[\int \left((X_i(\phi_i))^T i_0^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] - \vec{1}^T i_0^\partial H_i^{*\partial} \right) dt \right. \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \phi_i}, X_i(\phi_i) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_i^*, *_0 \vec{1}) \Big|_{t_i}^{t_f} \right] \\
& + \sum_j \left[\int (-1)^{n-1} \left((i_{n-1}^\partial (Y_j(\phi_j^*))^\partial)^T *_n^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - (i_{n-1}^\partial \vec{1})^T H_j \right) dt \right. \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_j, *_n^{-1} \vec{1}) \Big|_{t_i}^{t_f} \right].
\end{aligned}$$

And from Proposition 3.3.4 we have:

$$\begin{aligned}
0 = & \sum_i \left[\int \sum_{\sigma^0} \langle d_{n-1}^{\text{dual}} \left(X_i(\phi_i) \wedge *_1 \left(\frac{\delta S}{\delta d^0 \phi_i} \right) - \vec{1} \wedge H_i^* \right), *\sigma^0 \rangle dt \right. \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \phi_i}, X_i(\phi_i) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_i^*, *_0 \vec{1}) \Big|_{t_i}^{t_f} \right] \\
& + \sum_j \left[\int \sum_{\sigma^n} \langle d^{n-1} \left((-1)^{n-1} Y_j(\phi_j^*) \wedge *_n^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - H_j \wedge \vec{1} \right), \sigma^n \rangle dt \right. \\
& \left. + \left(\frac{\delta S}{\delta \partial_t \beta_j^*}, Y_j(\beta_j^*) \right) \Big|_{t_i}^{t_f} - (\tilde{H}_j, *_n^{-1} \vec{1}) \Big|_{t_i}^{t_f} \right]
\end{aligned}$$

Writing this with a total derivative in time gives:

$$\begin{aligned}
0 = & \sum_i \left[\int \sum_{\sigma^0} \left\{ \langle d_{n-1}^{\text{dual}} \left(X_i(\phi_i) \wedge *_1 \left(\frac{\delta S}{\delta d^0 \phi_i} \right) - \vec{1} \wedge H_i^* \right), *\sigma^0 \rangle \right. \right. \\
& \left. \left. + \langle \partial_t \left(X_i(\phi_i) \wedge *_0 \frac{\delta S}{\delta \partial_t \phi_i} - \tilde{H}_i \right), *\sigma^0 \rangle \right\} dt \right] \\
& + \sum_j \left[\int \sum_{\sigma^n} \left\{ \langle d^{n-1} \left((-1)^{n-1} Y_j(\phi_j^*) \wedge *_n^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - H_j \wedge \vec{1} \right), \sigma^n \rangle \right. \right. \\
& \left. \left. + \langle \partial_t \left(Y_j(\beta_j^*) \wedge *_n^{-1} \frac{\delta S}{\delta \partial_t \phi_j^*} - \tilde{H}_j \right), \sigma^n \rangle \right\} dt \right] \\
0 = & \sum_i \left[\int \sum_{\sigma^0} \left\{ \langle d_{n-1}^{\text{dual}} \left(X_i(\phi_i) \wedge *_1 \left(\frac{\delta S}{\delta d^0 \phi_i} \right) - H_i^* \right), *\sigma^0 \rangle \right. \right. \\
& \left. \left. + \langle \partial_t \left(X_i(\phi_i) \wedge *_0 \frac{\delta S}{\delta \partial_t \phi_i} - \tilde{H}_i \right), *\sigma^0 \rangle \right\} dt \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_j \left[\int \sum_{\sigma^n} \left\{ \langle d^{n-1} \left((-1)^{n-1} Y_j(\phi_j^*) \wedge *_{n-1}^{-1} \left(\frac{\delta S}{\delta d_0^{\text{dual}} \phi_j^*} \right) - H_j \right), \sigma^n \rangle \right. \right. \\
& \quad \left. \left. + \langle \partial_t \left(Y_j(\beta_j^*) \wedge *_{n-1}^{-1} \frac{\delta S}{\delta \partial_t \phi_j^*} - \tilde{H}_j \right), \sigma^n \rangle \right\} dt \right]
\end{aligned}$$

where in the last step we used that for a dual $(n-1)$ -cochain H_i^* and 0-cochain $\vec{1}$ that $\vec{1} \wedge H_i^* = H_i^*$ and likewise for a primal $(n-1)$ -cochain H_j and dual 0-cochain $\vec{1}$ that $H_j \wedge \vec{1} = H_j$. \square

Example 4.2.9. The symmetry for complex scalar field theory is the multiplication of ϕ by $e^{i\phi}$ where ϕ is a real number. Under this transformation our fields transform as:

$$\begin{aligned}
\phi & \mapsto \phi' = e^{i\phi} \phi \\
\bar{\phi} & \mapsto \bar{\phi}' = e^{-i\phi} \bar{\phi}.
\end{aligned}$$

Since $d^0 e^{i\alpha} \phi = e^{i\alpha} d^0 \phi$ and $\partial_t e^{i\alpha} \phi = e^{i\alpha} \partial_t \phi$, the action remains unchanged:

$$\begin{aligned}
S & \mapsto S' = \int \frac{1}{2} \left[(d^0 e^{i\alpha} \phi, d^0 e^{-i\alpha} \bar{\phi}) + (\partial_t e^{i\alpha} \phi, \partial_t e^{-i\alpha} \bar{\phi}) \right] + \frac{m}{2} (e^{i\alpha} \phi, e^{-i\alpha} \bar{\phi}) dt \\
& = \int \frac{1}{2} \left[(e^{i\alpha} d^0 \phi, e^{-i\alpha} d^0 \bar{\phi}) + (e^{i\alpha} \partial_t \phi, e^{-i\alpha} \partial_t \bar{\phi}) \right] + \frac{m}{2} (e^{i\alpha} \phi, e^{-i\alpha} \bar{\phi}) dt \\
& = \int \frac{1}{2} \left[(\partial_t \phi, \partial_t \bar{\phi}) + (d^0 \phi, d^0 \bar{\phi}) \right] + \frac{m}{2} (\phi, \bar{\phi}) dt,
\end{aligned}$$

where the last step follows from the definition of the real inner product. The conserved current is therefore:

$$\begin{aligned}
J^* & = i\phi \wedge *_1 \frac{\delta S}{\delta d^0 \phi} - i\bar{\phi} \wedge *_1 \frac{\delta S}{\delta d^0 \bar{\phi}} \\
& = i\phi \wedge *_1 \frac{1}{2} d^0 \bar{\phi} - i\bar{\phi} \wedge *_1 \frac{1}{2} d^0 \phi \\
& = \frac{i}{2} \left[\phi \wedge *_1 d^0 \bar{\phi} - \bar{\phi} \wedge *_1 d^0 \phi \right],
\end{aligned}$$

and the conserved charge is:

$$\begin{aligned}
Q^* & = i\phi \wedge *_0 \frac{\delta S}{\delta \partial_t \phi} - i\bar{\phi} \wedge *_0 \frac{\delta S}{\delta \partial_t \bar{\phi}} \\
& = i\phi \wedge *_0 \frac{1}{2} \partial_t \bar{\phi} - i\bar{\phi} \wedge *_0 \frac{1}{2} \partial_t \phi \\
& = \frac{i}{2} \left[\phi \wedge *_0 \partial_t \bar{\phi} - \bar{\phi} \wedge *_0 \partial_t \phi \right]
\end{aligned}$$

Example 4.2.10. The symmetry for O(n) field theory is similar to the scalar field theory. The symmetry transformation is multiplication of ϕ by R where R is a $n \times n$ real-rotation matrix. We

can always write that matrix as $R = e^r$, where r is the generator of the rotation. The field then transforms as:

$$\phi \mapsto \phi' = R\phi ,$$

and since $d^0 R\phi = R d^0 \phi$, the action remains unchanged:

$$\begin{aligned} S \mapsto S' &= \int \frac{1}{2} \left[(d^0 R\phi, d^0 R\phi) + (\partial_t R\phi, \partial_t R\phi) \right] + \frac{m}{2} (R\phi, R\phi) dt \\ &= \int \frac{1}{2} \left[(R d^0 \phi, R d^0 \phi) + (R \partial_t \phi, R \partial_t \phi) \right] + \frac{m}{2} (R\phi, R\phi) dt \\ &= \int \frac{1}{2} \left[(d^0 \phi, d^0 \phi) + (\partial_t \phi, \partial_t \phi) \right] + \frac{m}{2} (\phi, \phi) dt , \end{aligned}$$

where the last step follows from the definition of the inner product. The conserved current is now:

$$\begin{aligned} J^* &= \phi \wedge *_1 \frac{\delta S}{\delta d^0 \phi} \\ &= \phi \wedge *_1 d^0 \phi \\ &= \phi \wedge *_1 d^0 \phi , \end{aligned}$$

and the charge density is:

$$Q^* = \phi \wedge *_0 \partial_t \phi$$

4.2.3. Hamiltonian Formulation

Since we have split the time component out separately from the spatial components, the semi-discretization is a natural environment for the Hamiltonian formulation of mechanics. First recall that in the smooth setting the canonical momentum is defined as:

$$\Pi_\alpha = \frac{\delta L}{\delta \partial_t \alpha}$$

which inspires our discrete definition:

Definition 4.2.11. Given a semi-discrete field theory with action S and primal cochains α_i and α_i^t and dual cochains ϕ_j^* and β_j^{t*} , the canonical momentum to α_i is:

$$\Pi_{\alpha_i} := \frac{\delta S}{\delta \partial_t \alpha_i} ,$$

and likewise the canonical momentum to β_j^* is:

$$\Pi_{\beta_j^*} := \frac{\delta S}{\delta \partial_t \beta_j^*} .$$

Note that the fields α_i^t and β_i^t cannot have canonical momenta because they do not have time derivatives.

Definition 4.2.12. The **discrete Hamiltonian** is defined as:

$$\mathcal{H} := \int \left\{ \sum_i (\partial_t \alpha_i, \Pi_{\alpha_i}) + \sum_j (\partial_t \beta_j^*, \Pi_{\beta_j^*}) \right\} dt - S$$

Example 4.2.13. Recall that the action for the semi-discrete complex scalar field theory is:

$$S = \int \left\{ \frac{1}{2} [(\partial_t \phi, \partial_t \bar{\phi}) + (d^0 \phi, d^0 \bar{\phi})] + \frac{m}{2} (\phi, \bar{\phi}) \right\} dt ,$$

This leads to two conjugate momenta, one for ϕ and one for $\bar{\phi}$:

$$\begin{aligned} \Pi_\phi &= \frac{\delta S}{\delta \partial_t \phi} = \frac{1}{2} \partial_t \bar{\phi} \\ \Pi_{\bar{\phi}} &= \frac{\delta S}{\delta \partial_t \bar{\phi}} = \frac{1}{2} \partial_t \phi . \end{aligned}$$

With this we can easily compute the Hamiltonian:

$$\begin{aligned} \mathcal{H} &= \int \left\{ (\partial_t \phi, \Pi_\phi) + (\partial_t \bar{\phi}, \Pi_{\bar{\phi}}) \right\} dt - S \\ &= \int \frac{1}{2} (\partial_t \phi, \partial_t \bar{\phi}) + \frac{1}{2} (\partial_t \phi, \partial_t \bar{\phi}) - \left\{ \frac{1}{2} [(\partial_t \phi, \partial_t \bar{\phi}) + (d^0 \phi, d^0 \bar{\phi})] + \frac{m}{2} (\phi, \bar{\phi}) \right\} dt \\ &= \int \left\{ \frac{1}{2} [(\partial_t \phi, \partial_t \bar{\phi}) - (d^0 \phi, d^0 \bar{\phi})] - \frac{m}{2} (\phi, \bar{\phi}) \right\} dt \\ &= \int \left\{ \frac{1}{2} [(\Pi_\phi, \Pi_{\bar{\phi}}) - (d^0 \phi, d^0 \bar{\phi})] - \frac{m}{2} (\phi, \bar{\phi}) \right\} dt . \end{aligned}$$

Example 4.2.14. As before, $O(n)$ theory is similar to complex scalar field theory. First recall the action is:

$$S = \int \frac{1}{2} [(\partial_t \phi, \partial_t \phi) + (d^0 \phi, d^0 \phi)] + \frac{m}{2} (\phi, \phi) dt ,$$

where the inner product is as described in Example 4.1.2. This leads to n conjugate momenta, one for each component of ϕ :

$$\Pi_\phi = \frac{\delta S}{\delta \partial_t \phi} = \partial_t \phi .$$

The Hamiltonian also is similar to complex scalar field theory:

$$\begin{aligned} \mathcal{H} &= \int \left\{ (\partial_t \phi, \Pi_\phi) \right\} dt - S \\ &= \int (\partial_t \phi, \partial_t \phi) - \left\{ \frac{1}{2} [(\partial_t \phi, \partial_t \phi) + (d^0 \phi, d^0 \phi)] + \frac{m}{2} (\phi, \phi) \right\} dt \end{aligned}$$

$$\begin{aligned}
&= \int \left\{ \frac{1}{2} [(\partial_t \phi, \partial_t \phi) - (d^0 \phi, d^0 \phi)] - \frac{m}{2} (\phi, \phi) \right\} dt \\
&= \int \left\{ \frac{1}{2} [(\Pi_\phi, \Pi_\phi) - (d^0 \phi, d^0 \phi)] - \frac{m}{2} (\phi, \phi) \right\} dt .
\end{aligned}$$

Example 4.2.15. Abelian Yang-Mills is given by the action

$$S = \int \frac{1}{2} (\partial_t A, \partial_t A) + (d^0 A^t, \partial_t A) + \frac{1}{2} (d^0 A^t, d^0 A^t) + \frac{1}{2} (d^1 A, d^1 A) dt ,$$

Giving the conjugate momenta:

$$\Pi_A = \frac{\delta S}{\delta \partial_t A} = d^0 A^t + \partial_t A .$$

As well as the discrete Hamiltonian:

$$\begin{aligned}
\mathcal{H} &= \int (\partial_t A, \Pi_A) dt - S \\
&= \int (\partial_t A, d^0 A^t) + (\partial_t A, \partial_t A) dt \\
&\quad - \int \frac{1}{2} (\partial_t A, \partial_t A) + (d^0 A^t, \partial_t A) + \frac{1}{2} (d^0 A^t, d^0 A^t) + \frac{1}{2} (d^1 A, d^1 A) dt \\
&= \int \frac{1}{2} (\partial_t A, \partial_t A) - \frac{1}{2} (d^0 A^t, d^0 A^t) - \frac{1}{2} (d^1 A, d^1 A) dt \\
&= \int \frac{1}{2} (\Pi_A, \Pi_A) - (\Pi_A, d^0 A^t) + \frac{1}{2} (d^0 A^t, d^0 A^t) - \frac{1}{2} (d^1 A, d^1 A) dt
\end{aligned}$$

Chapter 5.

Discrete Gauge Theory and Yang-Mills

We develop discrete gauge theory by providing an extension of the standard covariant derivative which is defined on zero-cochains to a *discrete exterior derivative* which is defined for any vector-valued cochain. We use this definition to construct a discrete analog of curvature along with defining the wedge product for general $GL(n)$ -valued cochains and vector-valued cochains.

5.1. Connections and Parallel Transport

Connections take the place of exterior derivatives on sections in gauge theories and have been developed. We aim here to recall the definition for discrete connections given in Chapter 2 which was defined on primal 0-cochains and define an identical connection on dual 0-cochains. We then can prove a discrete Leibniz rule. Finally we show that, given a metric on the fibers of the discrete vector bundle, the connection “respects the metric.” We will develop the codifferential for our discrete connection later, after generalizing the connection on 0-cochains to the covariant exterior derivative which acts on any vector-valued k -cochain.

Definition 5.1.1. Recall given a vector bundle with parallel transport matrix U the **discrete primal connection** is defined as (Definition (2.5.1)):

$$\langle \nabla s, [v_0, v_1] \rangle := \langle U, [v_0, v_1] \rangle \langle s, [v_1] \rangle - \langle s, [v_0] \rangle ,$$

for any primal edge $[v_0, v_1]$. Likewise the **discrete dual connection** is defined as:

$$\langle \nabla s^*, [\star\sigma_0^n, \star\sigma_1^n] \rangle := \langle U, [\star\sigma_0^n, \star\sigma_1^n] \rangle \langle s^*, [\star\sigma_1^n] \rangle - \langle s^*, [\star\sigma_0^n] \rangle ,$$

for any dual edge $\star\sigma^1 = [\star\sigma_0^n, \star\sigma_1^n]$.

Note that just as a connection can be defined in terms of parallel transport, given a connection the parallel transport matrices are completely determined:

Proposition 5.1.2. *Given a discrete (primal) connection the parallel transport matrices are determined from the formula:*

$$\langle e_j^T U e_i, [v_0, v_1] \rangle := \langle e_j^T \nabla e_i \chi_{v_1}, [v_0, v_1] \rangle ,$$

where χ_{v_k} is the indicator function that takes the value 1 on vertex v_k and 0 elsewhere and e_i is the standard basis vector with 1 in slot i .

Proposition 5.1.3. ∇ as defined above satisfies the Leibniz rule. That is for any primal section s and primal 0-cochain f then on any primal edge $\sigma^1 = [v_i, v_j]$:

$$\langle \nabla(fs), [v_i, v_j] \rangle = \langle d f \wedge s + f \nabla s, [v_i, v_j] \rangle ,$$

where the wedge product is the primal-primal wedge from Definition (2.1.6).

Proof. We will show this by working out both sides of the above equality independently.

First consider $\langle \nabla(fs), [v_i, v_j] \rangle$:

$$\langle \nabla(fs), [v_i, v_j] \rangle = \langle U, [v_i, v_j] \rangle \langle f, [v_j] \rangle \langle s, [v_j] \rangle - \langle f, [v_i] \rangle \langle s, [v_i] \rangle .$$

Now for $\langle d f \wedge s + f \nabla s, [v_i, v_j] \rangle$:

$$\begin{aligned} \langle d f \wedge s + f \nabla s, [v_i, v_j] \rangle &= \frac{1}{2} (\langle f, [v_j] \rangle - \langle f, [v_i] \rangle) (\langle U, [v_i, v_j] \rangle \langle s, [v_j] \rangle + \langle s, [v_i] \rangle) \\ &\quad + \frac{1}{2} (f([v_j]) + f([v_i])) (U_{[v_i, v_j]} s([v_j]) - s([v_i])) \\ &= U_{[v_i, v_j]} f([v_j]) s([v_j]) - f([v_i]) s([v_i]) . \end{aligned}$$

□

Given a metric on the fibers, we can choose parallel transport matrices in $U(n)$ or $O(n)$. Then given a metric we would like our discrete connection to “respect the metric.”

Proposition 5.1.4. *Given a discrete vector bundle with metric and $U(n)$ structure group, the discrete connection respects the metric, i.e. for any pair of sections s and s' we have*

$$d^0 (s \cdot s') = \nabla s \cdot s' + s \cdot \nabla s' ,$$

Proof. We will work each side of the equality separately and show that these are equal. Let $\sigma^1 = [01]$ be a primal edge, expanding the left-hand side gives:

$$\langle d^0 (s \cdot s'), [01] \rangle = \langle \bar{s}, [1] \rangle \cdot \langle s', [1] \rangle - \langle \bar{s}, [0] \rangle \cdot \langle s', [0] \rangle .$$

And the right-hand side:

$$\begin{aligned}
\langle \nabla s \cdot s' + s \cdot \nabla s', [01] \rangle &= (\langle U, [01] \rangle \langle s, [1] \rangle - \langle s, [0] \rangle) \cdot \left(\frac{\langle U, [01] \rangle \langle s', [1] \rangle + \langle s', [0] \rangle}{2} \right) \\
&\quad + \left(\frac{\langle U, [01] \rangle \langle s, [1] \rangle + \langle s, [0] \rangle}{2} \right) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle - \langle s', [0] \rangle) \\
&= \frac{1}{2} (\langle U, [01] \rangle \langle s, [1] \rangle) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle) \\
&\quad - \frac{1}{2} (\langle s, [0] \rangle) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle) \\
&\quad + \frac{1}{2} (\langle U, [01] \rangle \langle s, [1] \rangle) \cdot \langle s', [0] \rangle \\
&\quad - \frac{1}{2} \langle s, [0] \rangle \cdot \langle s', [0] \rangle \\
&\quad + \frac{1}{2} (\langle U, [01] \rangle \langle s, [1] \rangle) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle) \\
&\quad + \frac{1}{2} (\langle s, [0] \rangle) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle) \\
&\quad - \frac{1}{2} (\langle U, [01] \rangle \langle s, [1] \rangle) \cdot \langle s', [0] \rangle \\
&\quad - \frac{1}{2} \langle s, [0] \rangle \cdot \langle s', [0] \rangle .
\end{aligned}$$

Since the two terms with only one parallel transport matrix in them occur twice and with opposite signs these cancel and since U is a special unitary matrix that means:

$$(\langle U, [01] \rangle \langle s, [1] \rangle) \cdot (\langle U, [01] \rangle \langle s', [1] \rangle) = \langle s, [1] \rangle \cdot \langle s', [1] \rangle .$$

And so the right-hand side simplifies to:

$$\begin{aligned}
\langle \nabla s \cdot s' + s \cdot \nabla s', [01] \rangle &= \langle s, [1] \rangle \cdot \langle s', [1] \rangle - \langle s, [0] \rangle \cdot \langle s', [0] \rangle \\
&= \langle d^0(s \cdot s'), [01] \rangle
\end{aligned}$$

□

Finally, we will be interested in endomorphism-valued cochains which will need to act on our vector-valued cochain. When this happens we obtain a new vector-valued cochain whose degree is the sum of the degrees of the same as the sum of the endomorphism-valued cochain and original vector-valued cochain.

Definition 5.1.5. Given a vector-valued k -cochain α and a endomorphism-valued l -cochain A we define the action of A on α as:

$$\langle A\alpha, [012...(k+l)] \rangle = \langle A, [01...l] \rangle \langle \alpha, [l(l+1)...(k+l)] \rangle .$$

5.1.1. Discrete Covariant Derivative: Existing Definition

We extend Definition (2.5.1) to a discrete covariant exterior derivative which acts on vector-valued 1-cochains. While this definition is able to reproduce a discrete analog of Definition (2.2.4) in terms of the discrete definition of curvature from holonomy (Definition (2.5.2)), we have not been able to extend this to general k -cochains. The common definition of curvature given in the literature is

$$\langle F, [012] \rangle = \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle - I .$$

Definition. The discrete covariant exterior derivative on vector-valued one-cochains $\alpha \in C^1(K, E)$ is:

$$\langle d^\nabla \alpha, [012] \rangle = \langle U, [01] \rangle \langle \alpha, [12] \rangle - \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \langle \alpha, [02] \rangle + \langle \alpha, [01] \rangle .$$

This definition is best explained by picture. In figure 5.1, the strategy of this definition is to always return a section to the lowest numbered vertex of the simplex, but every section needs to be “carried around” the entire simplex.

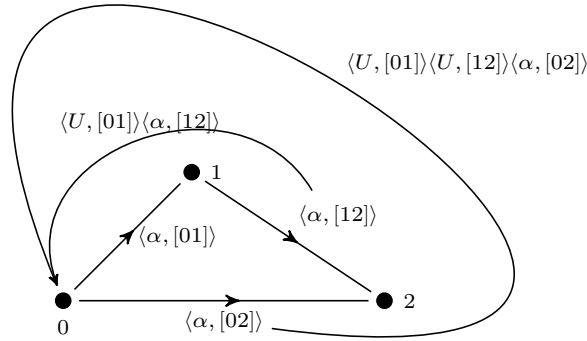


Figure 5.1.: “Carrying around” the various 1-cochains on a two-simplex. Not shown are the parallel transport matrices U of which there is one for each edge. The vertices are labeled 0, 1, and 2 for indexing. Note that $\langle \alpha, [01] \rangle$ is unchanged as it is already living in the fiber of vertex 0.

Proposition 5.1.6. Given a vector-valued 0-cochain s ,

$$\langle d^\nabla d^\nabla s, [012] \rangle = \langle F \wedge s, [012] \rangle .$$

Proof. Applying d^∇ to Definition (2.5.1) for the covariant derivative we find:

$$\begin{aligned} \langle d^\nabla(\nabla s), [012] \rangle &= \langle U, [01] \rangle \langle (\nabla s), [12] \rangle - \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \langle \nabla s, [02] \rangle + \langle \nabla s, [01] \rangle \\ &= \langle U, [01] \rangle (\langle U, [12] \rangle \langle s, [2] \rangle - \langle s, [1] \rangle) \\ &\quad - \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle (\langle U, [02] \rangle \langle s, [2] \rangle - \langle s, [2] \rangle) \end{aligned}$$

$$\begin{aligned}
& + (\langle U, [01] \rangle \langle s, [1] \rangle - \langle s, [0] \rangle) \\
& = \langle U, [01] \rangle \langle U, [12] \rangle \langle s, [2] \rangle - \langle U, [01] \rangle \langle s, [1] \rangle - \langle U, [01] \rangle \langle U, [12] \rangle \langle s, [2] \rangle \\
& \quad + \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \langle s, [0] \rangle + \langle U, [01] \rangle \langle s, [1] \rangle - \langle s, [0] \rangle \\
& = (\langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle - I) \langle s, [0] \rangle .
\end{aligned}$$

□

However, we have not been able to extend this definition to general vector-valued cochains. There are many different ways to bring a general k -cochain around the boundary of a $(k+1)$ -cell and we have not found an ordering that reproduces Proposition (2.2.7) from the introduction; that application of the covariant exterior derivative twice is the same as applying the curvature to the form. For example, simply following lexicographical does not give $d^\nabla d^\nabla = F \wedge \alpha$ for 3-cochain α .

5.1.2. Discrete Covariant Derivative: An Extension to Higher Cochains

We present a definition of the discrete covariant derivative, first presented in the context of synthetic differential geometry [Kock 1996], that is able to be extended to general vector-valued k -cochains as well allow use to prove discrete analogs of the smooth propositions listed in Section 2.2. This extension will however, require a different definition of discrete curvature. After discussing this definition and the intuition behind it, we will show analogs of smooth properties such as the Bianchi identity. We then also prove that $\langle d^\nabla d^\nabla \alpha, \sigma^{k+2} \rangle = \langle F \wedge \alpha, \sigma^{k+2} \rangle$ for any vector-valued k -cochain.

Definition 5.1.7. Given a discrete vector bundle $E \rightarrow M$ with parallel transport matrices U , the **discrete curvature** is an endomorphism-valued 2-cochain, $F \in C^2(K, \text{end}(E))$, defined on a two-cell $\sigma^2 = [012]$ by:

$$\langle F, [012] \rangle = \langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle$$

This definition should be interpreted as moving a section from the highest-numbered vertex of a triangle to the lowest along the two paths in the triangle. That is, as shown in Figure 5.2a, moving from vertex $2 \rightarrow 0$ via the paths $2 \rightarrow 1 \rightarrow 0$ as well as $2 \rightarrow 0$ and comparing how much the vector is changed under parallel transport. This is unlike the more common definition given in the literature (Definition (2.5.2)) which is the measure of how much a vector is changed when it is brought all of the way around a triangle back to its starting point (Figure 5.2b).

Although this definition of curvature is different than the standard definition, it transforms similarly under gauge transformation:

Proposition 5.1.8. *Given a discrete vector bundle with parallel transport matrices U and a gauge transformation g , the discrete curvature transforms as:*

$$\langle F, [012] \rangle \mapsto \langle g, [0] \rangle \langle F, [012] \rangle \langle g^{-1}, [2] \rangle$$

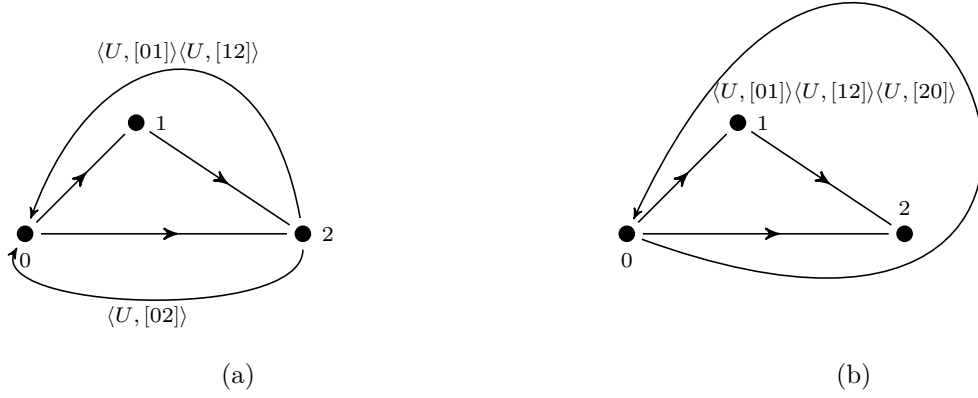


Figure 5.2.: (a) Schematic diagram visualizing the new definition of discrete curvature (Definition (5.1.7)). (b) Schematic diagram visualizing the more common definition of discrete curvature in the literature (Definition (2.5.2)).

Proof. We will work this out by direct computation. From the definition of discrete curvature we have:

$$\langle F, [012] \rangle = \langle U, 01 \rangle \langle U, 12 \rangle - \langle U, 02 \rangle .$$

Under a gauge transformation g , the parallel transport matrices U transform as:

$$\langle U, [01] \rangle \mapsto \langle g, [0] \rangle \langle U, [01] \rangle \langle g^{-1}, [1] \rangle ,$$

and so the curvature transforms as:

$$\begin{aligned} \langle F, [012] \rangle &\mapsto \langle g, [0] \rangle \langle U, 01 \rangle \langle g^{-1}, [1] \rangle \langle g, [1] \rangle \langle U, [12] \rangle \langle g^{-1}, [2] \rangle - \langle g, [0] \rangle \langle U, [02] \rangle \langle g^{-1}, [2] \rangle \\ &= \langle g, [0] \rangle \langle F, [012] \rangle \langle g^{-1}, [2] \rangle . \end{aligned}$$

□

Furthermore the discrete curvature behaves reasonably under changes of base point and under hermitian conjugation. Hermitian conjugation acts by changing the ordering of the vertices while preserving the orientation of the simplex, and the curvature after reordering the vertices of a simplex is related to the curvature before reordering through parallel transport as the following Proposition shows.

Proposition 5.1.9. *Given a discrete vector bundle with curvature F , on any two simplex σ^2 , the discrete curvature has the following properties:*

1. $\langle F^\dagger, [012] \rangle = \langle F, [210] \rangle$
2. $\langle F, [102] \rangle = -\langle U, [10] \rangle \langle F, [012] \rangle$

$$3. \langle F, [021] \rangle = -\langle F, [012] \rangle \langle U, [21] \rangle$$

$$4. \langle F, [120] \rangle = \langle U, [10] \rangle \langle F, [012] \rangle \langle U, [20] \rangle$$

$$5. \text{ If } g \text{ are gauge transformations then: } \langle F, [012] \rangle \mapsto \langle g, [0] \rangle \langle F, [012] \rangle \langle g^{-1}, [2] \rangle$$

Proof. 1. $\langle F^\dagger, [012] \rangle = \langle F, [210] \rangle$

By definition:

$$\begin{aligned} \langle F^\dagger, [012] \rangle &= (\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle)^\dagger \\ &= \langle U, [21] \rangle \langle U, [10] \rangle - \langle U, [02] \rangle \\ &= \langle F, [210] \rangle . \end{aligned}$$

$$2. \langle F, [102] \rangle = -\langle U, [10] \rangle \langle F, [012] \rangle$$

Again using the definition of discrete curvature:

$$\begin{aligned} \langle F, [102] \rangle &= \langle U, [10] \rangle \langle U, [02] \rangle - \langle U, [12] \rangle \\ &= \langle U, [10] \rangle (\langle U, [02] \rangle - \langle U, [01] \rangle \langle U, [12] \rangle) \\ &= -\langle U, [10] \rangle (\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle) \\ &= -\langle U, [10] \rangle \langle F, [012] \rangle . \end{aligned}$$

$$3. \langle F, [120] \rangle = \langle U, [10] \rangle \langle F, [012] \rangle \langle U, [20] \rangle$$

The definition of discrete curvature gives:

$$\begin{aligned} \langle F, [120] \rangle &= \langle U, [12] \rangle \langle U, [20] \rangle - \langle U, [10] \rangle \\ &= \langle U, [10] \rangle (\langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle - 1) \\ &= \langle U, [10] \rangle (\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle) \langle U, [20] \rangle \\ &= \langle U, [10] \rangle \langle F, [012] \rangle \langle U, [20] \rangle . \end{aligned}$$

□

Definition 5.1.10. Given a vector valued primal k -cochain, $\alpha \in C^k(K, E)$ on a discrete vector space with parallel transport matrices U the **discrete covariant exterior covariant** is defined on a $(k+1)$ -cochain $\sigma = [0123\dots k]$ by:

$$\langle d^\nabla \alpha, [0123\dots k] \rangle := \langle U, [01] \rangle \langle \alpha, [123\dots k] \rangle + \sum_{j=1}^k (-1)^j \langle \alpha, [01\dots \hat{j} \dots k] \rangle$$

Note that on sections, $s \in C^0(K, E)$:

$$\langle d^\nabla s, [01] \rangle = \langle U, [01] \rangle \langle s, [1] \rangle - \langle s, [0] \rangle$$

which matches Definition (2.5.1) for the covariant derivative. We can also generalize this definition to act on endomorphism valued forms by:

Definition 5.1.11. The extension of the discrete covariant exterior derivative to an endomorphism-valued k -cochain A is:

$$\begin{aligned} \langle d_k^\nabla A, [012\dots(k+1)] \rangle &:= \langle U, [01] \rangle \langle A, [12\dots(k+1)] \rangle \\ &+ \sum_{j=1}^k \left[(-1)^j \langle A, [01\dots\hat{j}\dots(k+1)] \rangle \right] \\ &+ (-1)^{k+1} \langle A, [01\dots k] \rangle \langle U, [k(k+1)] \rangle . \end{aligned}$$

Theorem 5.1.12. *The discrete curvature satisfies a Bianchi identity:*

$$d_2^\nabla F = 0$$

Proof. We will show this on an arbitrary three-simplex $\sigma^3 = [0123]$:

$$\langle d_2^\nabla F, [0123] \rangle = \langle U, [01] \rangle \langle F, [123] \rangle - \langle F, [023] \rangle + \langle F, [013] \rangle - \langle F, [012] \rangle \langle U, [23] \rangle$$

Expanding the curvature using Definition (5.1.7):

$$\begin{aligned} \langle d_2^\nabla F, [0123] \rangle &= \langle U, [01] \rangle (\langle U, [12] \rangle \langle U, [23] \rangle - \langle U, [13] \rangle) \\ &\quad - (\langle U, [02] \rangle \langle U, [23] \rangle - \langle U, [03] \rangle) \\ &\quad + (\langle U, [01] \rangle \langle U, [13] \rangle - \langle U, [03] \rangle) \\ &\quad - (\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [12] \rangle) \langle U, [23] \rangle \\ &= 0 \end{aligned}$$

□

Recall from Chapter 2, that in the smooth setting for any vector valued differential form α we have that acting by the covariant exterior derivative twice gives is the same as acting on form with the curvature. Our primal discrete covariant exterior derivative satisfies a discrete version of that property where the action of an endomorphism-valued cochain on a vector-valued cochain is as defined in Definition (5.1.5).

Proposition 5.1.13. *Given a discrete vector bundle with connection and a vector-valued $(k-1)$ -cochain α then*

$$\langle d^\nabla d^\nabla \alpha, \sigma^{k+1} \rangle = \langle F\alpha, \sigma^{k+1} \rangle$$

Proof. Applying the covariant exterior derivative twice to a primal vector-valued $(k-1)$ -cochain α gives:

$$\begin{aligned}
\langle d^\nabla d^\nabla \alpha, [01\dots k+1] \rangle &= \langle U, [01] \rangle \langle d^\nabla \alpha, [12\dots k+1] \rangle + \sum_{i=1}^{k+1} (-1)^i \langle d^\nabla \alpha, [01\dots \hat{i} \dots k+1] \rangle \\
&= \langle U, [01] \rangle \left(\langle U, [12] \rangle \langle \alpha, [23\dots k+1] \rangle + \sum_{i=2}^{k+1} (-1)^{i-1} \langle \alpha, [12\dots \hat{i} \dots k+1] \rangle \right) \\
&\quad - \langle U, [02] \rangle \langle \alpha, [23\dots k+1] \rangle + \sum_{i=2}^{k+1} (-1)^i \langle U, [01] \rangle \langle \alpha, [12\dots \hat{i} \dots k+1] \rangle \\
&\quad + \sum_{i=1}^{k+1} \left[\sum_{\substack{j=1 \\ j < i}}^{k+1} (-1)^{i+j} \langle \alpha, [01\dots \hat{j} \dots \hat{i} \dots k+1] \rangle \right. \\
&\quad \left. + \sum_{\substack{j=1 \\ j > i}}^{k+1} (-1)^{i+j-1} \langle \alpha, [01\dots \hat{i} \dots \hat{j} \dots k+1] \rangle \right] \\
&= \langle U, [01] \rangle \langle U, [12] \rangle \alpha[23\dots k+1] - \langle U, [02] \rangle \langle \alpha, [23\dots k+1] \rangle \\
&= \langle F\alpha, \sigma^{k+1} \rangle .
\end{aligned}$$

where in the last step we applied Definition (5.1.5). \square

Proposition 5.1.14. *Given a discrete primal vector-valued k -cochain α and an endomorphism-valued l -cochain A there is a discrete Leibniz rule:*

$$\begin{aligned}
\langle d^\nabla (A\alpha), [01\dots k+l+1] \rangle &= \langle d^\nabla A, 01\dots l+1 \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
&\quad + (-1)^l \langle A, 01\dots l \rangle \langle d^\nabla \alpha, [l\dots k+l+1] \rangle .
\end{aligned}$$

Proof.

$$\begin{aligned}
\langle d^\nabla (A\alpha), [01\dots k+l+1] \rangle &= \langle U, [01] \rangle \langle A \wedge \alpha, [12\dots k+1] \rangle + \sum_{i=1}^{k+l+1} \langle A \wedge \alpha, [01\dots \hat{i} \dots k+l] \rangle \\
&= \langle U, [01] \rangle \langle A, [12\dots l+1] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
&\quad + \sum_{i=1}^l (-1)^i \langle A, [01\dots \hat{i} \dots l+1] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
&\quad + \sum_{i=l+1}^{k+l+1} (-1)^i \langle A, [01\dots l] \rangle \langle \alpha, [l\dots \hat{i} \dots k+l+1] \rangle \\
&= \langle U, [01] \rangle \langle A, [12\dots l+1] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
&\quad + \sum_{i=1}^l (-1)^i \langle A, [01\dots \hat{i} \dots l+1] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle
\end{aligned}$$

$$\begin{aligned}
& + (-1)^l \langle A, [01\dots l] \rangle \langle U, [l(l+1)] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
& - (-1)^l \langle A, [01\dots l] \rangle \langle U, [l(l+1)] \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
& + \sum_{i=l+1}^{k+l+1} (-1)^i \langle A, [01\dots l] \rangle \langle \alpha, [l\dots \hat{i}\dots k+l+1] \rangle \\
& = \langle d^\nabla A, 01\dots l+1 \rangle \langle \alpha, [l+1\dots k+l+1] \rangle \\
& + (-1)^l \langle A, 01\dots l \rangle \langle d^\nabla \alpha, [l\dots k+l+1] \rangle
\end{aligned}$$

□

5.2. Dual Discrete Covariant Exterior Derivative

In the next three sections, we extend the proofs of Chapter 4 from the discrete exterior derivative to the connection as well as prove discrete analogs of many smooth properties of the covariant exterior derivative to the dual exterior derivative which is complicated by the additional boundary terms. Recall for primal vector-valued cochains that we identified the fiber of the lowest-numbered vertex as the fiber for that cochain. For vector-valued dual cochains we also will use the fiber of the lowest *primal* vertex as the fiber for the vector-valued dual cochain. This allows us to define the dual discrete covariant derivative in terms of the primal discrete covariant derivative.

Definition 5.2.1. The **dual discrete covariant exterior derivative** is defined as:

$$\left(d_{n-k}^\nabla\right)^{\text{dual}} := \left[(-1)^k \left(d_{k-1}^\nabla\right)^\dagger, (-1)^{k-1} i_{k-1}^\partial \right]$$

We have again added an additional term to the discrete covariant derivative, just as we did to the discrete dual exterior derivative (Section 3.1) to “close the loop” of the dual cell that intersects the boundary. Our definition of discrete curvature was dependent on the simplicial structure of the primal mesh which we do not have in the dual mesh. Instead, we will use the smooth Proposition 2.2.7 to define the dual curvature. This carries over to the definition of dual curvature where the definition is given as:

Definition 5.2.2. The **dual curvature** is defined by applying the dual discrete covariant exterior derivative twice. That is, given a vector-valued dual $(n-k)$ -cochain α^* :

$$F^* \alpha^* := \left(d_{n-k-1}^\nabla\right)^{\text{dual}} \left(d_{n-k}^\nabla\right)^{\text{dual}} \alpha^*.$$

5.3. Adjointness of the Covariant Exterior Derivative

Following the examples in Chapter 3 we define the primal and dual codifferential as:

Definition 5.3.1. The **primal discrete connection codifferential** is defined as:

$$\left(d_k^\nabla\right)^* \alpha := (-1)^{n(k-1)+1} *_k^{-1} \left(d_{n-k}^\nabla\right)^{\text{dual}} \begin{bmatrix} *_k \alpha \\ \text{tr} * \alpha \end{bmatrix}.$$

The **dual discrete connection codifferential** is defined as:

$$\left(d_{n-k}^\nabla\right)_{\text{dual}}^* \beta^* = (-1)^{(k+1)(n-k)} *_{k+1} d_k^\nabla *_k^{-1} \beta^*.$$

As in the case of the exterior derivative, the codifferential for the connection is its adjoint.

Proposition 5.3.2.

$$\left(d_{k-1}^\nabla \beta, \alpha\right) - \left(\beta, \left(d_k^\nabla\right)^* \alpha\right) = \beta^T i_{k-1}^\partial \text{tr} * \alpha,$$

for any primal k -cochain α and primal $(k-1)$ -cochain β .

Proof. Consider the inner product:

$$\begin{aligned} \left(\beta, \left(d_k^\nabla\right)^* \alpha\right) &= \left(\beta, (-1)^{n(k-1)+1} *_k^{-1} \left((-1)^k \left(d_{k-1}^\nabla\right)^\dagger *_k \alpha + (-1)^{k-1} i_{k-1}^\partial \text{tr} * \alpha\right)\right) \\ &= (-1)^{nk-n+1+k} \beta^T *_k^{-1} (-1)^k *_k^{-1} \left(\left(d_{k-1}^\nabla\right)^\dagger *_k \alpha - i_{k-1}^\partial \text{tr} * \alpha\right) \\ &= (-1)^{nk-n+1+k+(k-1)(k-1)} \left(\beta^T \left(d_{k-1}^\nabla\right)^\dagger *_k \alpha - \beta^T i_{k-1}^\partial \text{tr} * \alpha\right) \\ &= \left(d_{k-1}^\nabla \beta, \alpha\right) - \beta^T i_{k-1}^\partial \text{tr} * \alpha. \end{aligned}$$

□

Proposition 5.3.3. The dual discrete exterior derivative and dual codifferential are adjoints. That is, given a dual $(n-k)$ -cochain α and dual $(n-k-1)$ -cochain β :

$$\left(\left(d_{n-k-1}^\nabla\right)^{\text{dual}} \beta, \alpha\right) - \left(\beta, \left(d_{n-k}^\nabla\right)_{\text{dual}}^* \alpha\right) = (-1)^k \left(i_k^\partial \beta^\partial\right)^T *_k^{-1} \alpha.$$

Proof.

$$\begin{aligned} \left(\left(d_{n-k-1}^\nabla\right)^{\text{dual}} \beta, \alpha\right) &= \left((-1)^{k+1} \left(d_k^\nabla\right)^\dagger \beta + (-1)^k i_k^\partial \beta^\partial\right)^T *_k^{-1} \alpha \\ &= \beta^\dagger (-1)^{k+1} d_k^\nabla *_k^{-1} \alpha + \left(i_k^\partial \beta^\partial\right)^T (-1)^k *_k^{-1} \alpha \\ &= \beta^T (-1)^{k+1+(k+1)(n-k-1)} *_k^{-1} *_k^{-1} d_k^\nabla *_k^{-1} \alpha + \left(i_k^\partial \beta^\partial\right)^T (-1)^k *_k^{-1} \alpha \\ &= \beta^T *_k^{-1} (-1)^{(k+1)(n-k)} *_k^{-1} d_k^\nabla *_k^{-1} \alpha + \left(i_k^\partial \beta^\partial\right)^T (-1)^k *_k^{-1} \alpha \\ &= \beta^T *_k^{-1} \left(d_{n-k}^\nabla\right)^* \alpha + \left(i_k^\partial \beta^\partial\right)^T (-1)^k *_k^{-1} \alpha \\ &= \left(\beta, \left(d_{n-k}^\nabla\right)^* \alpha\right) + \left(i_k^\partial \beta^\partial\right)^T (-1)^k *_k^{-1} \alpha. \end{aligned}$$

□

5.3.1. Euler-Lagrange Equations

We are now in a position to define the Euler-Lagrange equations for theories involving a connection. While in the smooth setting, often these can be derived from the usual Euler-Lagrange equations, in the discrete setting it is more natural to derive these from scratch. The proof will proceed very similarly to what was derived in Chapter 4. We begin, by varying the fields α_i and β_j^* :

$$\alpha_i \mapsto \alpha_i + \delta\alpha_i \quad \text{and} \quad \beta_j^* \mapsto \beta_j^* + \delta\beta_j^* .$$

This induces the variations on the derivatives:

$$d_{k_i}^\nabla \alpha_i \mapsto d_{k_i}^\nabla \alpha_i + d_{k_i}^\nabla \delta\alpha_i \quad \text{and} \quad (d_{n-k_j}^\nabla)^\text{dual} \beta_j^* \mapsto (d_{n-k_j}^\nabla)^\text{dual} \beta_j^* + (d_{n-k_j}^\nabla)^\text{dual} \delta\beta_j^* .$$

This imposes variations on S :

$$\begin{aligned} \delta S = & \sum_{k=0}^n \left(\frac{\delta S}{\delta \alpha_i}, \delta\alpha_i \right) + \left(\frac{\delta S}{\delta d_{k_i}^\nabla \alpha_i}, d_{k_i}^\nabla \delta\alpha_i \right) \\ & + \left(\frac{\delta S}{\delta \beta_j^*}, \delta\beta_j^* \right) + \left(\frac{\delta S}{\delta ((d_{n-k_j}^\nabla)^\text{dual} \beta_j^*)}, (d_{n-k_j}^\nabla)^\text{dual} \delta\beta_j^* \right) . \end{aligned}$$

We will now derive that the Euler-Lagrange equations are stationary points of the action. We first vary the action with respect to perturbations of the fields. Then we apply adjointness of d and d^* . Finally, we need to regroup terms. We begin by varying the action with respect to perturbations

$$\alpha_i \mapsto \alpha_i + \delta\alpha_i \quad \text{and} \quad \beta_j^* \mapsto \beta_j^* + \delta\beta_j^* ,$$

and find that:

$$\begin{aligned} \delta S = & \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta\alpha_i \right) + \left(\frac{\delta S}{\delta d_{k_i}^\nabla \alpha_i}, d_{k_i}^\nabla \delta\alpha_i \right) \right] \\ & + \sum_j \left[\left(\frac{\delta S}{\delta \beta_j^*}, \delta\beta_j^* \right) + \left(\frac{\delta S}{\delta ((d_{n-k_j}^\nabla)^\text{dual} \beta_j^*)}, (d_{n-k_j}^\nabla)^\text{dual} \delta\beta_j^* \right) \right] . \end{aligned}$$

We can now apply adjointness of d and d^* . As usual we will not allow our fields to vary on the boundary and so no boundary terms will appear.

$$\delta S = \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i}, \delta\alpha_i \right) + \left((d_{k_i+1}^\nabla)^* \left(\frac{\delta S}{\delta d_{k_i}^\nabla \alpha_i} \right), \delta\alpha_i \right) \right]$$

$$+ \sum_j \left[\left(\frac{\delta S}{\delta \beta_k}, \delta \beta_k \right) + \left((d_{n-k_j-1}^\nabla)^* \left(\frac{\delta S}{\delta \left((d_{n-k_j}^\nabla)^{\text{dual}} \beta_j^* \right)} \right), \delta \beta_j^* \right) \right] .$$

Combining the terms gives:

$$\begin{aligned} \delta S = & \sum_i \left[\left(\frac{\delta S}{\delta \alpha_i} + (d_{k_i+1}^\nabla)^* \left(\frac{\delta S}{\delta d_{k_i}^\nabla \alpha_i} \right), \delta \alpha_i \right) \right] \\ & + \sum_j \left[\left(\frac{\delta S}{\delta \beta_k} + (d_{n-k_j-1}^\nabla)^{\text{dual}} \left(\frac{\delta S}{\delta \left((d_{n-k_j}^\nabla)^{\text{dual}} \beta_j^* \right)} \right), \delta \beta_k \right) \right] . \end{aligned}$$

Since this must be zero for any variations $\delta \alpha_i$ and $\delta \beta_k$ we have the **discrete Euler-Lagrange Equations**:

$$\frac{\delta S}{\delta \alpha_i} + (d_{k_i+1}^\nabla)^* \left(\frac{\delta S}{\delta d_{k_i}^\nabla \alpha_i} \right) = 0 \quad (5.1)$$

$$\frac{\delta S}{\delta \beta_j^*} + (d_{n-k_j-1}^\nabla)^{\text{dual}} \left(\frac{\delta S}{\delta \left((d_{n-k_j}^\nabla)^{\text{dual}} \beta_j^* \right)} \right) = 0 \quad (5.2)$$

5.4. Discrete Yang-Mills

We now have all of the ingredients for a discrete Yang-Mills theory. The discrete Yang-Mills action is given by:

$$S = \text{tr} (F, F) + \text{tr} (A, J) := \text{tr} [F^\dagger *_2 F] + \text{tr} [A^\dagger *_1 J] ,$$

where F is the curvature and A is gauge field defined as the solution to $d_1^\nabla A = F$.

The Euler-Lagrange equation is then given by:

$$\frac{\delta S}{\delta A} + (d_2^\nabla)^* \left(\frac{\delta S}{\delta F} \right) = 0 ,$$

which yields:

$$J + (d_2^\nabla)^* F = 0 .$$

Recalling the Bianchi identity gives the discrete equations of motion for the Yang-Mills action:

$$(d_2^\nabla)^* F = J$$

$$d_2^\nabla F = 0 .$$

This framework also provides a discrete method to examine soliton and self-adjoint solutions which are defined by:

$$F = *_2 F$$

Because the discrete curvature satisfies the Bianchi identity, we know that these are solutions to the equations of motion without current:

$$\begin{aligned} 0 &= d_2^\nabla F \\ &= d_2^\nabla *_2 F \quad \Rightarrow \quad (d_2^\nabla)^* F = 0 . \end{aligned}$$

However since the DEC hodge star is diagonal, this is only possible if $*_2 = \mathbb{1}$. An interesting approach would be to consider a doubled curvature, one on the primal mesh, F , and one on the dual mesh, F^* , in that way $F^* = *_2 F$ could be defined between these two endomorphism value cochains.

5.5. Example: Complex $U(1)$ Field Theory

We consider the example of a $U(1)$ gauge field coupled to a bosonic field. While it would be more physical to couple the gauge field to a spinor field, developing spinor fields remains outside the scope of this thesis. However, we note that there are no major obstructions to the development of discrete spinor fields. The gauge-invariant action for discrete complex scalar field theory is given by:

$$S = \frac{1}{2} (d_0^\nabla \phi, d_0^\nabla \bar{\phi}) + m(\phi, \bar{\phi}) - \frac{1}{2} (F, F) ,$$

where ϕ is a complex 0-cochain and $\bar{\phi}$ is its complex conjugate and as before F is the curvature associated to the connection d_0^∇ . Note that this is a generalization of the complex scalar field theory given in Chapter 4, in which the connection was taken to be the trivial connection.

The Euler-Lagrange equations for this system involve three separate equations one for ϕ , $\bar{\phi}$ and F :

$$\begin{aligned} \frac{\delta S}{\delta \phi} + (d_1^\nabla)^* \left(\frac{\delta S}{\delta d_0^\nabla \phi} \right) &= 0 \\ \frac{\delta S}{\delta \bar{\phi}} + (d_1^\nabla)^* \left(\frac{\delta S}{\delta d_0^\nabla \bar{\phi}} \right) &= 0 \\ (d_2^\nabla)^* \left(\frac{\delta S}{\delta F} \right) &= 0 \end{aligned}$$

The Euler-Lagrange equation for F is the same one we derived for the Yang-Mills Theory above and note that the Euler-Lagrange equations for ϕ and $\bar{\phi}$ are complex conjugates of each other. Both are needed because the two real components of the complex field need to be allowed to vary independently (see Example 4.1.1) for more details. To derive the conserved currents we will vary the action with respect to the gauge transformation:

$$\begin{aligned}\phi &\mapsto \phi' = e^{if}\phi \\ \bar{\phi} &\mapsto \bar{\phi}' = e^{-if}\bar{\phi} \\ U &\mapsto U' = e^{if}Ue^{-ig}\end{aligned}$$

Under these transformations the action varies as:

$$\begin{aligned}\delta S &= \left(\frac{\delta S}{\delta \phi}, \delta \phi \right) + \left(\frac{\delta S}{\delta d_0^\nabla \phi}, d_0^\nabla \delta \phi \right) + \left(\frac{\delta S}{\delta \bar{\phi}}, \delta \bar{\phi} \right) + \left(\frac{\delta S}{\delta d_0^\nabla \bar{\phi}}, d_0^\nabla \delta \bar{\phi} \right) \\ &\quad + \left(\frac{\delta S}{\delta A}, \delta A \right) + \left(\frac{\delta S}{\delta d_1^\nabla A}, d_1^\nabla \delta A \right) .\end{aligned}$$

Note that varying U induces transformations on A . We can insert the Euler-Lagrange equations into this and obtain:

$$\begin{aligned}\delta S &= \left((d_1^\nabla)^* \left(\frac{\delta S}{\delta \phi} \right), \delta \phi \right) + \left(\frac{\delta S}{\delta d_0^\nabla \phi}, d_0^\nabla \delta \phi \right) + \left((d_1^\nabla)^* \left(\frac{\delta S}{\delta \bar{\phi}} \right), \delta \bar{\phi} \right) + \left(\frac{\delta S}{\delta d_0^\nabla \bar{\phi}}, d_0^\nabla \delta \bar{\phi} \right) \\ &\quad + \left((d_2^\nabla)^* \left(\frac{\delta S}{\delta A} \right), \delta A \right) + \left(\frac{\delta S}{\delta d_1^\nabla A}, d_1^\nabla \delta A \right) .\end{aligned}$$

Inserting in the form of the variations we obtain:

$$\begin{aligned}\delta S &= \left((d_1^\nabla)^* \left(\frac{\delta S}{\delta \phi} \right), if\phi \right) + \left(\frac{\delta S}{\delta d_0^\nabla \phi}, d_0^\nabla (if\phi) \right) + \left((d_1^\nabla)^* \left(\frac{\delta S}{\delta \bar{\phi}} \right), -if\bar{\phi} \right) \\ &\quad + \left(\frac{\delta S}{\delta d_0^\nabla \bar{\phi}}, d_0^\nabla (-if\bar{\phi}) \right) + \left((d_2^\nabla)^* \left(\frac{\delta S}{\delta A} \right), \delta A \right) + \left(\frac{\delta S}{\delta d_1^\nabla A}, d_1^\nabla \delta A \right) .\end{aligned}$$

This leads us to obtain only boundary terms for the ϕ and $\bar{\phi}$ fields:

$$\delta S = (if \wedge \phi)^T i_0^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] + (-if \wedge \bar{\phi})^T i_0^\partial \text{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] + \left(\frac{\delta S}{\delta A}, -d^0 f \right) .$$

Re-grouping these we have:

$$\begin{aligned}
\delta S &= (f)^T \left(i\phi \wedge i_0^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] \right) + (f)^T \left(-i\bar{\phi} \wedge i_0^\partial \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \bar{\phi}} \right) \right] \right) \\
&\quad + \left(\frac{\delta S}{\delta A}, -d^0 f \right) \\
&= (f)^T i_0^\partial \left(i\phi \wedge \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \phi_i} \right) \right] \right) + (f)^T i_0^\partial \left(-i\bar{\phi} \wedge \operatorname{tr} \left[* \left(\frac{\delta S}{\delta d^0 \bar{\phi}} \right) \right] \right) \\
&\quad + \left(\frac{\delta S}{\delta A}, -d^0 f \right) ,
\end{aligned}$$

where in the second equality we have used the co-dimension one primal-dual wedge product defined in Definition (3.3.3). In analogy to complex scalar field theory we can define the particle current density as:

$$J^* = \frac{i}{2} \left[\phi \wedge *_1 \left(\frac{\delta S}{\delta d^0 \phi} \right) - \bar{\phi} \wedge *_1 \left(\frac{\delta S}{\delta d^0 \bar{\phi}} \right) \right] .$$

Plugging this into our expression for the variation of the action we have:

$$\begin{aligned}
\delta S &= (f)^T i_0^\partial J^\partial + \left(\frac{\delta S}{\delta A}, -d^0 f \right) \\
&= \sum_{*\sigma^0} \langle d_{n-1}^{\text{dual}} (f \wedge J), *\sigma^0 \rangle + \left(\frac{\delta S}{\delta A}, -d^0 f \right) \\
&= \sum_{*\sigma^1} \langle d^0 f \wedge J, V_{*\sigma^1} \rangle + \sum_{\sigma^0} \langle f \wedge d_{n-1}^{\text{text}} J, V_{*\sigma^0} \rangle + \left(\frac{\delta S}{\delta A}, -d^0 f \right) \\
&= \sum_{*\sigma^1} \langle d^0 f \wedge J, V_{*\sigma^1} \rangle + \left(\frac{\delta S}{\delta A}, -d^0 f \right) \\
&= (d^0 f, *_1^{-1} J) + \left(-d^0 f, \frac{\delta S}{\delta A} \right) .
\end{aligned}$$

where we have used $\sum_{\sigma^0} \langle d_{n-1}^{\text{dual}} J^*, *\sigma^0 \rangle = 0$. The above expression is zero for any arbitrary f if:

$$*_1 \frac{\delta S}{\delta A} = J ,$$

and so we define the gauge current as:

Definition 5.5.1. The **gauge current** is defined as:

$$J_{\text{gauge}}^* = *_1 \frac{\delta S}{\delta A} .$$

5.6. Numerical Experiments on Scalar Field Theory

Consider the semi-discrete massless complex scalar field theory whose discrete action is given by:

$$S = \frac{1}{2} \int dt [(\partial_t \phi, \partial_t \bar{\phi}) + (d_0^\nabla \phi, d_0^\nabla \bar{\phi})]$$

The Euler-Lagrange equations for this theory are:

$$\begin{aligned} \frac{\delta S}{\delta \phi} - \partial_t \left(\frac{\delta S}{\delta \partial_t \phi} \right) + (d_1^\nabla)^* \left(\frac{\delta S}{\delta d_0^\nabla \phi} \right) &= 0 \\ \frac{\delta S}{\delta \bar{\phi}} - \partial_t \left(\frac{\delta S}{\delta \partial_t \bar{\phi}} \right) + (d_1^\nabla)^* \left(\frac{\delta S}{\delta d_0^\nabla \bar{\phi}} \right) &= 0. \end{aligned}$$

Giving the equations of motion:

$$\begin{aligned} -\partial_t^2 \phi + (d_1^\nabla)^* d_0^\nabla \phi &= 0 \\ -\partial_t^2 \bar{\phi} + (d_1^\nabla)^* d_0^\nabla \bar{\phi} &= 0. \end{aligned}$$

We will assume our solutions are stationary solutions in time which means they time dependence is of the form e^{-iEt} , where E is the energy. This gives ϕ and $\bar{\phi}$:

$$\begin{aligned} \phi &= e^{-iEt} \phi \\ \bar{\phi} &= e^{iEt} \bar{\phi}, \end{aligned}$$

where θ is a purely discrete zero-cochain and $\bar{\theta}$ its complex conjugate. Plugging into the Euler-Lagrange equations gives:

$$\begin{aligned} -(d_1^\nabla)^* d_0^\nabla \phi &= E\phi \\ -(d_1^\nabla)^* d_0^\nabla \bar{\phi} &= E\bar{\phi}. \end{aligned}$$

We computed the energy eigenvalues of this theory on the unit sphere for the Levi-Civita connection and compared the results to the known spectrum in Table 5.1.

Eigenvalue	N_2				Exact Value
	328	1218	2624	7922	
λ_0	0.9906	0.9975	0.9956	0.9985	1
λ_1	0.9999	0.9996	0.9996	0.9996	1
λ_2	1.007	1.003	1.003	1.000	1
λ_3	4.824	4.949	4.965	4.974	5
λ_4	4.846	4.973	4.982	4.983	5
λ_5	4.896	4.979	4.985	4.989	5
λ_6	4.927	4.981	4.995	5.006	5
λ_7	5.021	4.991	5.001	5.012	5
λ_8	10.20	10.81	10.85	10.94	11
λ_9	10.27	10.82	10.88	10.95	11

Table 5.1.: Comparison of the spectrum of the complex scalar field coupled to a Levi-Civita gauge field on the sphere with the known values. Note that every eigenvalue is doubled what is shown here, one for the θ and one for $\bar{\theta}$; for compactness only one is shown.

Chapter 6.

Discrete Abelian BF Theory

We consider both a primal and a dualized version of two different discretizations of Abelian BF theory in three spacetime dimensions. Throughout this chapter our actions and many of the other quantities will be written in terms of Hodge stars which are not typically used in BF theories. These are the topological Hodge stars (2.1.11) and simply serve to move information between the primal and dual complexes.

6.1. Semi-Discrete Action

We present a discretization of abelian BF theory cellular complexes embedded on closed Riemann surfaces. For this theory there is no restriction on the graphs, besides that they be planar. We will show that our discretization has a discrete analog of the smooth equations of motion by using the results of Chapter 4 as well as discrete versions of the following properties of the smooth theory:

1. Gauge invariance,
2. For two paths the commutator of the Wilson loops around the paths is proportional to the (oriented) intersection number, and
3. Demonstrate local flux attachment.

Recall that BF theory describes the dynamics of the connection one-form A , which we will write as $A = A_t dt + A_x dx$, where dx represents all of the spatial directions. We semi-discretize this by sampling the time components A_t at both the primal and dual vertices. We also integrate the spatial components on both the primal and dual edges. We will denote the dual components with asterisks. As mentioned in the introduction to this chapter, the Hodge stars that appear are the topological Hodge stars (2.1.11) which identify maps between the primal k -cochains and dual $(n - k)$ -cochains, while the inner products are given by the vector dot product (2.1.8). Our semi-discrete actions are then given as:

$$S^{\text{primal}} = \frac{k}{2\pi} \int dt \left[(A_t^*, *_2 d^1 A_x) + (A_x^*, *_1 d^0 A_t) - (A_x^*, *_1 \dot{A}_x) \right]$$

$$S^{\text{dual}} = \frac{k}{2\pi} \int dt \left[(A_t, *_0^{-1} d_1^{\text{dual}} A_x^*) + (A_x, *_1^{-1} d_2^{\text{dual}} A_t^*) - (A_x, *_1^{-1} \dot{A}_x^*) \right] .$$

In S^{primal} A_\bullet^* is the discrete analog of B , while $d^1 A_x$ and \dot{A}_x are the components of the discretization of F . While in S^{dual} A_{bullet} is the discrete analog of B , while $d_1^{\text{dual}} A_x^*$ and \dot{A}_x^* are the components of the discretization of F . We note that this action is local, that is only gauge fields that intersect interact. Furthermore, as $t \rightarrow \pm\infty$ we will require that $A \rightarrow 0$ and $A^* \rightarrow 0$. We will be especially interested in the average of S^{primal} and S^{dual} which we will simply denote as S . Explicitly, S is given by:

$$S = \frac{k}{2\pi} \int dt \left[(A_t^*, *_2 d^1 A_x) + (A_t, *_0^{-1} d_1^{\text{dual}} A_x^*) - \frac{1}{2} \left[(A_x, *_1^{-1} \dot{A}_x^*) + (A_x^*, *_1 \dot{A}_x) \right] \right] ,$$

where we have used the adjointness of d and d^* to reduce terms. As we will show in Subsection 6.1.2 this combined action is gauge invariant while the primal and dual actions are not. In addition, it is self dual. This means that under a duality transformation, one that exchanges primal and dual meshes, the action is unchanged.

6.1.1. Euler-Lagrange Equations

From Equations (4.3) - (4.6) the actions each have four equations of motion:

$$\begin{aligned} \frac{\delta S^\bullet}{\delta A_x} - \partial_t \left(\frac{\delta S^\bullet}{\delta \partial_t A_x} \right) + d_2^* \left(\frac{\delta S^\bullet}{\delta d^1 A_x} \right) &= 0 \\ \frac{\delta S^\bullet}{\delta A_t} + d_1^* \left(\frac{\delta S^\bullet}{\delta d^0 A_t} \right) &= 0 \\ \frac{\delta S^\bullet}{\delta A_x^*} - \partial_t \left(\frac{\delta S^\bullet}{\delta d_1^{\text{dual}} A_x^*} \right) + (d_1^{\text{dual}})^* \left(\frac{\delta S^\bullet}{\delta d_1^{\text{dual}} A_x^*} \right) &= 0 \\ \frac{\delta S^\bullet}{\delta A_t^*} + (d_1^{\text{dual}})^* \left(\frac{\delta S^\bullet}{\delta d_0^{\text{dual}} A_t^*} \right) &= 0 , \end{aligned}$$

where S^\bullet stands for S^{primal} , S^{dual} , or S . Working with Sp the four equations are:

$$\begin{aligned} -\frac{1}{2} \dot{A}_x^* &= 0 \\ d_1^{\text{dual}} A_x^* &= 0 \\ -\frac{1}{2} \partial_t A_x + (d_1^{\text{dual}})^* A_t &= 0 \\ d_1^{\text{dual}} A_x &= 0 . \end{aligned}$$

Identifying A^* as B and $d_1^{\text{dual}} A_x$ as F we see we obtain the usual equations of motion for BF theory. Working with S^{dual} gives a similar result with A and A^* exchanged. Due to the symmetry between primal and dual fields in S we will only work out the primal equations. Beginning with the top

equation:

$$\begin{aligned}
0 &= -\frac{1}{2} *_1^{-1} \dot{A}_x^* - \partial_t \left(\frac{1}{2} *_1^{-1} A_x^* \right) + d_2^* (A_t^*) \\
&= *_1^{-1} \dot{A}_x^* + d_2^* *_0^{-1} A_t^* \\
&= *_1^{-1} \dot{A}_x^* + *_1^{-1} d_0^{\text{dual}} A_t^* .
\end{aligned} \tag{6.1}$$

And now for the other primal equation:

$$0 = *_2 d^1 A_x . \tag{6.2}$$

Put together these equations imply that the total curvature is zero. Put more physically, the electric field can be identified with Equation (6.1), while the magnetic with Equation (6.2). Then these equations tell us that the stationary points of the action are when there is no electric or magnetic fields - that is no curvature.

6.1.2. Gauge Invariance

As we are considering surfaces without boundary, our BF functional should be gauge invariant. In this theory, a gauge transformation is a real-valued zero cochain applied at all of the primal (ϕ) and dual (ϕ^*) zero-cells. The gauge fields then transform as:

$$\begin{aligned}
A_t &\mapsto A_t - \dot{\phi} , \\
A_x &\mapsto A_x - d^0 \phi , \\
A_t^* &\mapsto A_t^* - \dot{\phi}^* , \\
A_x^* &\mapsto A_x^* - d_0^{\text{dual}} \phi^* .
\end{aligned}$$

Since gauge transformations represent duplicate descriptions of the same theory, the action should be unaffected by gauge transformations. While the primal and dual actions are individually not gauge invariant their sum is as we find in the following theorem.

Theorem 6.1.1. *The semi-discrete actions S^{primal} and S^{dual} are gauge invariant.*

Proof. Under a gauge transformation we have:

$$\begin{aligned}
S^{\text{primal}} &= \frac{k}{2\pi} \int dt \left\{ \left((A_t^* - \dot{\phi}^*), *_2 d^1 A_x \right) + \left((A_x^* - d_0^{\text{dual}} \phi^*), *_1 (d^0 A_t - d^0 \dot{\phi}) \right) \right. \\
&\quad \left. - \left((A_x^* - d_0^{\text{dual}} \phi^*), *_1 (\dot{A}_x - (d^0 \dot{\phi})) \right) \right\} \\
&= \frac{k}{2\pi} \int dt \left\{ (A_t^*, *_2 d^1 A_x) - (\dot{\phi}^*, *_2 d^1 A_x) \right. \\
&\quad \left. + (A_x^*, *_0 d^0 A_t) - (d_2^{\text{dual}} \phi^*, *_0 d^0 A_t) - (A_x^*, *_0 d^0 \dot{\phi}) + (d_2^{\text{dual}} \phi^*, *_1 d^0 \dot{\phi}) \right\}
\end{aligned}$$

$$\begin{aligned}
& - \left(A_x^*, *_1 \dot{A}_x \right) + \left(A_x^*, *_1 d^0 \dot{\phi} \right) + \left(d_2^{\text{dual}} \phi^*, *_1 \dot{A}_x \right) - \left(d_2^{\text{dual}} \phi^*, *_1 d^0 \dot{\phi} \right) \} \\
= & \frac{k}{2\pi} \int dt \left\{ \left(A_t^*, *_2 d^1 A_x \right) + \left(A_x^*, *_0 d^0 A_t \right) - \left(A_x^*, *_1 \dot{A}_x \right) \right. \\
& \left. - \left(\dot{\phi}^*, *_2 d^1 A_x \right) - \left(A_x^*, *_0 d^0 \dot{\phi} \right) + \left(A_x^*, *_1 d^0 \dot{\phi} \right) + \left(d_2^{\text{dual}} \phi^*, *_1 \dot{A}_x \right) \right\} .
\end{aligned}$$

where in the last step we adjointness of d and d^* and that $d_{\bullet}^{\text{dual}} = \pm d^T$ in for topological hodge stars, so that $d_{\bullet}^* d^{\bullet} = 0$. Finally, integrating by parts in time and using adjointness of d and d^* allows us to cancel the remaining terms:

$$\begin{aligned}
\int dt \left(d_2^{\text{dual}} \phi^*, *_1 \dot{A}_x \right) &= - \int dt \left(d_2^{\text{dual}} \dot{\phi}^*, *_1 A_x \right) \\
&= - \int dt \left(\dot{\phi}^*, (d_1^{\text{dual}})^* *_1 \dot{A}_x \right) \\
&= \int dt \left(\dot{\phi}^*, *_2 d^1 \dot{A}_x \right)
\end{aligned}$$

The calculation for S^{dual} is simply the dualized version of the calculation for S^{primal} and won't be repeated here. \square

Corollary 6.1.2. *The symmetrized action is gauge-invariant.*

Following our discussion of symmetry and conservation laws for semi-discrete theories in Chapter 4, we can define a discrete charge density as well as a discrete current density. Again we will use S^{\bullet} to denote S^{primal} , S^{dual} , or S .

Definition 6.1.3. The **primal (dual) discrete charge density** defined as:

$$\begin{aligned}
\rho &= \frac{\delta S^{\bullet}}{\delta A_t} \\
\rho^* &= \frac{\delta S^{\bullet}}{\delta A_t^*} .
\end{aligned}$$

Definition 6.1.4. The **primal (dual) discrete current density** is defined as:

$$\begin{aligned}
J &= \frac{\delta S^{\bullet}}{\delta A_x} \\
J^* &= \frac{\delta S^{\bullet}}{\delta A_x^*} .
\end{aligned}$$

Using these definitions, we can easily the functional derivative with respect to A_t :

$$\begin{aligned}
\left(\frac{\delta S^{\bullet}}{\delta A_t}, f \right) &= \frac{k}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{\left((A_t + \epsilon f), *_0^{-1} (d_1^{\text{dual}} A_x^*) \right) - \left(A_t, *_0^{-1} (d_1^{\text{dual}} A_x^*) \right)}{\epsilon} \\
&= \frac{k}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{\left(f, *_0^{-1} d_1^{\text{dual}} A_x^* \right)}{\epsilon}
\end{aligned}$$

$$= \frac{k}{2\pi} \left(f, *_0^{-1} d_1^{\text{dual}} A_x^* \right) .$$

Giving us the charge density:

$$\rho = \frac{k}{2\pi} \left(*_0^{-1} d_1^{\text{dual}} A_x^* \right) .$$

A similar calculation where we take the functional derivative with respect to A_t^* gives the dual charge density:

$$\rho^* = \frac{k}{2\pi} *_2 d^1 A_x .$$

Taking the functional derivative with respect to A_x gives:

$$\begin{aligned} \left(\frac{\delta S}{\delta A_x}, \eta \right) &= -\frac{k}{4\pi} \left\{ \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left[\left(\dot{A}_x^*, *_1 (A_x + \epsilon \eta), \right) - \left(A_x^*, *_1^{-1} \dot{A}_x \right) \right] \right. \\ &\quad \left. + \left[\left((A_x + \eta), \dot{A}_x^* \right) - \left(A_x^*, \dot{A}_x \right) \right] \right\} \\ &= -\frac{k}{2\pi} \left[\left(\dot{A}_x^*, *_1 \eta \right) + \left(\dot{A}_x, *_1^{-1} \eta \right) \right] \\ &= -\frac{k}{2\pi} \left(*_1^{-1} \dot{A}_x^*, \eta \right) , \end{aligned}$$

where the last equality is by definition of the inner product of dual cochains. Identifying like terms gives the primal current density:

$$J = -\frac{k}{2\pi} *_1^{-1} \dot{A}_x^* .$$

And similarly for the dual case we have:

$$J^* = -\frac{k}{2\pi} *_1 \dot{A}_x .$$

While the symmetrized action, S has both J and J^* as well as both ρ and ρ^* . S^{primal} has J^* and ρ^* (and not J and ρ), while S^{dual} as J and ρ (but not J^* and ρ^*). We expect the charge density and current density to not be unrelated, but instead that build up of charge density on a zero cell come from the flow of current density across its faces. This is the essence of the following proof about conservation of charge.

Proposition 6.1.5. *The primal discrete charge and current satisfy a conservation law:*

$$\dot{\rho} + d_1^* J = 0 ,$$

and similarly for the dual charge and current density:

$$\dot{\rho}^* + \left(d_1^{\text{dual}} \right)^* J^* = 0 .$$

Proof. The proofs for the two cases are similar and are by direct computation; therefore we only give the primal proof:

$$\begin{aligned}
\langle d_1^* J, \sigma^0 \rangle &= \langle *_0^{-1} d_1^{\text{dual}} *_1 J, \sigma^0 \rangle \\
&= \langle d_1^{\text{dual}} *_1 J, \star \sigma^0 \rangle \\
&= \frac{k}{2\pi} \langle -d_1^{\text{dual}} \dot{A}_x^*, \star \sigma^0 \rangle \\
&= \frac{k}{2\pi} \langle -*_0^{-1} d_1^{\text{dual}} \dot{A}_x^*, \sigma^0 \rangle \\
&= \frac{k}{2\pi} \partial_t \langle -d_1^{\text{dual}} A_x^*, \sigma^0 \rangle \\
&= -\langle \dot{\rho}, \sigma^0 \rangle .
\end{aligned}$$

□

Note that the primal and dual charge densities do not interact. This matches what we saw in the equations of motion (Equations (6.1) and 6.2). In fact, the electric and magnetic fields we saw in the equations of motion is related to these charge densities. Flux attachment is the requirement that magnetic flux must come from a charged particle. In the continuum the magnetic field from a charge located at r' is given by:

$$B(r) = \frac{2\pi q}{k} \delta(r - r') , \quad (6.3)$$

which shows that flux is coincident with the charge. This equation enforces the requirement that physical states be invariant under constant-time gauge transformations [Dirac 1966]. Recognizing $(d_1^{\text{dual}} A_x^*)$ as the magnetic flux, Φ through the dual region $\star \sigma^0$, we can rewrite this as:

$$\langle \Phi^*, \star \sigma^0 \rangle = \frac{2\pi}{k} \langle \rho^*, \star \sigma^0 \rangle ,$$

which is the integrated version of the smooth condition (6.1.2), instead of localized charges, we have a charge density and the net flux comes from the flux density.

6.1.3. Quantization and Commutation Relations

This subsection will focus on the symmetrized action, S , as this is the only theory that will have non-trivial commutation relations. Recall from Chapter 4 that the canonical momentum to a gauge field is defined as:

$$\Pi_A = \frac{\delta S}{\delta \dot{A}(x)} ,$$

where the derivative is the functional derivative. In the canonical quantization we have the canonical commutation relation:

$$[A(x'), \Pi_A(x)] = i\hbar \delta(x - x') .$$

In our case we have:

$$\begin{aligned}
(\Pi_A, \eta) &= \frac{k}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{(A_x^*, *_1^{-1} (\dot{A}_x + \epsilon \eta)) - (A_x^*, *_1^{-1} \dot{A}_x)}{\epsilon} \\
&= \frac{k}{2\pi} \lim_{\epsilon \rightarrow 0} \frac{(A_x^*, *_1^{-1} \epsilon \eta)}{\epsilon} \\
&= \frac{k}{2\pi} (A_x^*, *_1 \eta) .
\end{aligned}$$

where after matching like terms gives us:

$$\Pi_A = \frac{k}{2\pi} *_1^{-1} A_x^* .$$

And similarly for the dual case:

$$\Pi_{A^*} = \frac{k}{2\pi} *_1 A_x .$$

Note that S^{primal} has Π_A and S^{dual} as Π_{A^*} , which S has both. Recall that in the smooth setting, each field has a canonical momenta and since the canonical momenta to the field is proportional to the field itself there is an even number of independent degrees of freedom. Let σ_1^1 and σ_2^1 be one-cells. The discrete canonical commutation relations are then given by:

$$[\langle A_x, \sigma_1^1 \rangle, \langle \Pi_{A_x}, \sigma_2^1 \rangle] = \begin{cases} i\hbar & \text{if } \sigma_2^1 = \star \sigma_1^1 \\ 0 & \text{else} \end{cases} ,$$

and similarly for the dual relation. Since Π_{A_x} is proportional to A_x^* we can rewrite these relations as:

$$[\langle A_x, \sigma_1^1 \rangle, \langle A_x^*, \sigma_2^1 \rangle] = \frac{2\pi i\hbar}{k} \hat{\delta}_{\star \sigma_1^1, \sigma_2^1} ,$$

where $\hat{\delta}_{\star \sigma_1^1, \sigma_2^1}$ is a signed delta function. It is $+1$ if $\star \sigma_1^1 = \sigma_2^1$ with orientation and -1 if $\star \sigma_1^1 = \sigma_2^1$ but has opposite orientation. Note that this means that the commutator of the gauge fields is $\frac{2\pi i\hbar}{k}$ if the intersection of the edges is “right-handed” (see Figure 6.1) and $-\frac{2\pi i\hbar}{k}$ if the intersection is “left-handed.”

There is a similar relation in [Sun, Kumar, and Fradkin 2015] since the canonical momentum to the gauge field is proportional to the gauge field. They find that

$$[\langle A_e, \sigma_2^1 \rangle, \langle A_e, \sigma_2^1 \rangle] = \frac{2\pi i\hbar}{k} K_{\sigma_2^1, \sigma_1^1}^{-1} .$$



Figure 6.1.: (a) An example of a “right-handed” intersection. (b) an example of a “left-handed” intersection. In both the black line represents a primal path and a blue line represents a dual path. The line that is drawn unbroken is the one that appears first in the commutator while the broken edge represents the second in the commutator. These names come from the right hand rule. If the cross product of the two edges gives a vector that matches the ambient orientation of the space then the intersection is “right-handed,” otherwise it is “left-handed.”

6.1.4. Wilson Loops

Since we have a commutator between gauge fields, we can discuss the commutator of Wilson loops. Since we are dealing with an abelian theory, instead of Definition (2.4.8), we will use

$$W_P = \sum_{\sigma^1 \in P} (A_x)$$

as the definition of the discrete Wilson loop. This is simply the logarithm of Definition (2.4.8). In the discrete theory, paths are defined by a series of vertices $\{v_1, v_2, \dots, v_k\}$ with $[v_i, v_{i+1}]$ being an edge in the mesh. A loop is a path with $v_1 = v_k$. Unlike in the smooth theory, where small perturbations in the paths can transform non-transverse intersections into transverse intersections, small perturbations in the path are not possible in the discrete theory. The result is that some non-transverse intersections become ambiguous, dependent on infinitesimal details that cannot be resolved by the discretization, as shown in Figure 6.2.

We will define the discrete analog of the commutator of Wilson loops to only be defined between a loop in the primal and loop in the dual mesh.

Proposition 6.1.6. *Let P be a loop in the primal (dual) mesh and P' a loop in the dual (primal) mesh then:*

$$[W_P, W_{P'}] = \frac{2\pi i \hbar}{k} \nu[P, P'] ,$$

where $\nu[P, P']$ is the number of signed intersections of the loops.

Proof. We prove the case that if P is a loop in the primal mesh and P' is a loop in the dual mesh,

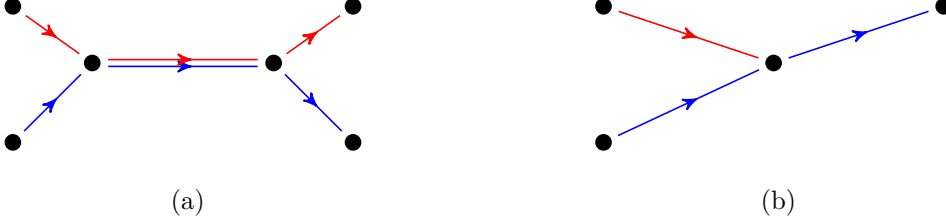


Figure 6.2.: Two examples of non-transverse intersections. In (a) this intersection can be unambiguously assigned the number $\nu = 1$ since any small perturbation of these paths would yield no or an equal number of positive and negative intersections. In (b), where the red path terminates at the center vertex there is no unambiguous intersection number to assign since in the smooth setting the value would be dependent on the infinitesimal details that the discrete model cannot resolve.

the other case is similar:

$$\begin{aligned}
[W_P, W_{P'}] &= \left[\sum_{\sigma^1 \in P} (A_x), \sum_{(\sigma^1)' \in P'} \langle A_x^*, (\sigma^1)' \rangle \right] \\
&= \sum_{\sigma^1 \in P} \sum_{(\sigma^1)' \in P'} \frac{2\pi i}{\hbar} \hat{\delta}_{\star \sigma^1, (\sigma^1)'} \\
&= \frac{2\pi i}{\hbar} \nu[P, P'] .
\end{aligned}$$

□

This result is highly dependent on the homotopy-type of the loop in question. If the path is contractible, for instance, the commutator must be zero.

Corollary 6.1.7. *If P or P' is contractible, then $[W_P, W_{P'}] = 0$.*

Proof. If a loop is contractible, then it must cross the other loop an even number of times with an equal number of left and right handed intersections. □

6.1.5. Consistency

An important test of such theories is consistency, which is the condition that the fluxes should commute. Recall that the flux of our semi-discrete BF theory is:

$$\Phi = \frac{k}{2\pi} *_2 d^1 A_x \quad \text{and} \quad \Phi^* = \frac{k}{2\pi} *_0^{-1} d_1^{\text{dual}} A_x^* ,$$

on primal and dual cells respectively. Since primal and dual fields commute with themselves we have that:

$$[\Phi, \Phi] = 0 \quad \text{and} \quad [\Phi^*, \Phi^*] = 0 .$$

Likewise if the primal and dual patches do not overlap the commutation is also zero trivially. When the patches overlap the commutation of the two intersecting primal and dual fields exactly cancel due to the relative orientations of the primal and dual edges, see Figure 6.3.

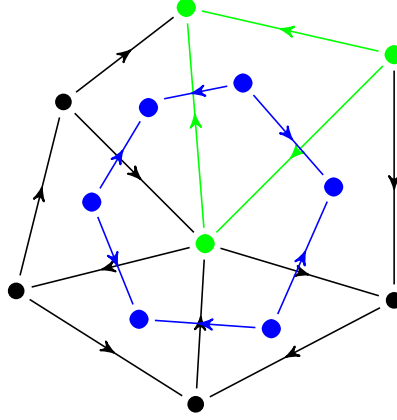


Figure 6.3.: Intersection of two regions of flux. The primal region is outlined in green and the dual region in blue. The edges where the boundary of the regions intersect give opposite orientations of the dual edges in the loop. The same would be true if one of the primal edges was oriented oppositely since the corresponding dual edge would also change orientations.

6.2. Fully Discrete Action

We construct a fully-discrete space-time action on a triangulated three-dimensional spacetime. We will place the integrated components of A on both the primal and dual edges. We then have the discrete action:

$$S^{\text{primal}} = \frac{k}{2\pi} (A^*, *_2(d^1 A))$$

$$S^{\text{dual}} = \frac{k}{2\pi} (A, *_1^{-1}(d_1^{\text{dual}} A^*)) .$$

Note, for example, that in the first term above A^* is on dual edges and $d^1 A$ is on primal triangles so the wedge product gives a three-dimensional object which is evaluated on the three-dimensional support volume of triangles. Thus the action above is a sum over three-dimensional objects that tile spacetime. As with the semi-discrete action these are named by where the discrete analog of the F field is placed. For example, in S^{primal} A^* is the discretization of B , while $d^1 A$ is the discretization of F . We will also consider the symmetrized action:

$$S = \frac{k}{4\pi} \left[(A^*, *_2(d^1 A)) + (A, *_1^{-1}(d_1^{\text{dual}} A^*)) \right] .$$

6.2.1. Euler-Lagrange Equations

From Equations (4.1) and (4.2) we see both S^{primal} and S^{dual} system has two Euler-Lagrange equations, one for the primal fields A and one for the dual fields A^* . For example, for S^{primal} :

$$\begin{aligned}\frac{\delta S^{\text{primal}}}{\delta A} + d_2^* \left(\frac{\delta S^{\text{primal}}}{\delta d^1 A} \right) &= 0 \\ \frac{\delta S^{\text{primal}}}{\delta A^*} + d_2^* \left(\frac{\delta S^{\text{primal}}}{\delta d^1 A^*} \right) &= 0 .\end{aligned}$$

Explicitly working out these equations we have:

$$\begin{aligned}\frac{k}{2\pi} d_2^* *_2^{-1} A^* &= \frac{k}{2\pi} *_1^{-1} d_1^{\text{dual}} A^* = 0 \\ \frac{k}{2\pi} *_2 d^1 A &= 0 .\end{aligned}$$

Identifying $d^1 A$ as the curvature F , and A^* as B we have:

$$\begin{aligned}d_1^{\text{dual}} B &= 0 \\ F &= 0 .\end{aligned}$$

Likewise for S^{dual} we have:

$$\begin{aligned}\frac{k}{2\pi} d^1 A &= 0 \\ \frac{k}{2\pi} *_2 d_1^{\text{dual}} A^* &= 0 .\end{aligned}$$

In this equation we identify $d_1^{\text{dual}} A^*$ as F^* and A as B , which gives the equations of motion for BF theory. Finally for S we again have two equations of motion. Due this action's more symmetrized form these equations are identical under interchange A and A^* . Explicitly the equation for A is:

$$\begin{aligned}0 &= \frac{k}{4\pi} d^1 A + \frac{k}{4\pi} d^1 (A) \\ &= \frac{k}{2\pi} d^1 A \\ &= F ,\end{aligned}$$

where following the discussion in Section 5.1.1 we have identified $d A$ with curvature.

6.2.2. Gauge Invariance

A gauge transformation is a choice of function ϕ for each vertex and ϕ^* for each dual vertex. The vector potentials A and A^* transform as:

$$\begin{aligned} A &\mapsto A - d^0 \phi \\ A^* &\mapsto A^* - d_1^{\text{dual}} \phi^* . \end{aligned}$$

Theorem 6.2.1. *The boundaryless fully-discrete action is gauge invariant.*

Proof.

$$\begin{aligned} S &= \frac{k}{4\pi} \left((A^* - d_0^{\text{dual}} \phi^*), (d^1 A) \right) + \left((A - d^0 \phi), (d_1^{\text{dual}} A^*) \right) \\ &= \left(A^*, (d^1 A) \right) + \left(d_0^{\text{dual}} \phi^*, (d^1 A) \right) \\ &\quad + \left(A, (d_1^{\text{dual}} A^*) \right) - \left((d^0 \phi), (d_1^{\text{dual}} A^*) \right) \\ &= \left(A^*, (d^1 A) \right) + \left(A, (d_1^{\text{dual}} A) \right) , \end{aligned}$$

where the last equality is due to Proposition (3.3.2) □

6.2.3. Charge Conservation

We will define the discrete primal and dual current densities in accordance with Definition (5.5.1):

$$\begin{aligned} J &= \frac{\delta S^{\text{dual}}}{\delta A} \\ J^* &= \frac{\delta S^{\text{primal}}}{\delta A^*} . \end{aligned}$$

As we discussed in Chapter 4 for fully-discrete actions we cannot easily identify the charge and current densities separately since most edges will be made of edges that have both space- and time-like character. These current can, however, be easily computed:

$$\begin{aligned} J &= \frac{k}{4\pi} *_1^{-1} d_1^{\text{dual}} A^* \\ J^{\text{dual}} &= \frac{k}{4\pi} *_2 d^1 A . \end{aligned}$$

Conservation of charge is also easier to prove in this context:

Proposition 6.2.2. *Charge is conserved, that is:*

$$\begin{aligned} d_1^* J &= 0 \\ \left(d_1^{\text{dual}} \right)^* J^* &= 0 , \end{aligned}$$

for primal and dual current densities, respectively

Proof. Indeed because $d_2^{\text{dual}} d_1^{\text{dual}} = 0$ we have:

$$d_1^* J = *_0^{-1} d_2^{\text{dual}} *_1 \left(\frac{k}{4\pi} *_1^{-1} d_1^{\text{dual}} A^* \right) = 0 .$$

and likewise for the dual current density. □

The symmetrized action S has both primal and dual currents which are independently conserved.

Chapter 7.

Characteristic Classes and Topological Charge

Topological charges play an important role in exotic condensed matter systems such as those involving dislocations and other attributes that cannot be smoothly deformed away. The most fundamental topological charge comes from the first Chern class. After providing a definition of the discrete first Chern class we prove that it obeys key properties from the smooth setting including under Whitney sum and tensor product, as well as showing that this definition is closed and will always integrate to an integer. We then compare our definition to two definitions inspired by our discrete curvature. We then discuss generalizations to the remaining Chern classes. We give four possible definitions, and demonstrate which properties these definitions lack.

7.1. Bundle Operations

We construct the discrete Whitney, tensor, and pullback bundles which will be necessary for discussing discretizations of Characteristic classes later.

Definition 7.1.1. Given a vector bundle $E \rightarrow K$ with connection and structure group G , a **subbundle** is a choice of subspace $U \subset V$ such that $\langle U, [v_j, v_i] \rangle \in H$ where H is a subgroup of G .

We are also interested in how to create a new vector bundle out of two (or more) vector bundles. These follow the constructions that are common for finite dimensional vector spaces namely the direct sum and tensor product. We only give the definition for two vector bundles, but this can be scaled to any (finite) number by using the universal properties of these operations.

Definition 7.1.2. Given two vector bundles with connection $E \rightarrow K$ and $E' \rightarrow K$ the **Whitney sum** of vector bundles is the vector bundle with fibers $E_{v_i} \oplus E'_{v_i}$ and parallel transport matrices that are direct sum of the corresponding parallel transport matrices.

Definition 7.1.3. Given two vector bundles with connection $E \rightarrow K$ and $E' \rightarrow K$ the **tensor product** of vector bundles is the vector bundle with fiber $E_{v_i} \otimes E'_{v_i}$ and parallel transport matrices given by tensor product of the two parallel transport matrices.

Definition 7.1.4. Let $E \rightarrow L$ be a vector bundle with connection. And $f : K \rightarrow L$ be a simplicial map. The **pullback bundle** is a vector bundle $E' \rightarrow K$ that makes the following diagram commute:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ K & \xrightarrow{f} & L \end{array} \quad \text{where the top arrow is an isomorphism on the fibers.}$$

7.2. First Chern Class

The first Chern class is a topological invariant that appears prominently in low-dimensional systems. In fact, it completely characterizes complex line bundles on surfaces up to isomorphism class.

Recall that Chern classes can be defined in terms of curvature using Chern-Weil theory. However, using the curvature defined in the previous section to define characteristic classes this way yields classes that are not-gauge invariant. While, this can be corrected for, our numerical experiments reveal that these definitions still lead to computational errors.

In [Phillips 1985], a definition of first Chern class is given in terms of holonomy for $U(1)$ bundles and bundles that are Whitney sums of $U(1)$ bundles. We use this definition for all structure groups and prove it is closed under the discrete exterior derivative as well as prove it behaves analogously as the smooth first Chern class under tensor product and Whitney sum.

Definition 7.2.1. The **first Chern class** of a vector bundle with connection is defined as :

$$\langle c_1, \sigma^2 \rangle = \frac{i}{2\pi} \log (\det (\sigma^2)) .$$

Note that this definition make sense for both primal and dual complexes. Also note that this is not the trace of the definition of discrete curvature (Definition 5.1.7) since this quantity would not be gauge-invariant:

$$\begin{aligned} \langle \text{tr } F, [012] \rangle &= \text{tr} [\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle] \\ &\mapsto \text{tr} [\langle g, [0] \rangle (\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle) \langle g^{-1}, [2] \rangle] \\ &\neq \text{tr} [\langle U, [01] \rangle \langle U, [12] \rangle - \langle U, [02] \rangle] . \end{aligned}$$

Proposition 7.2.2. c_1 is gauge invariant.

Proof. Under gauge transformation holonomy transforms as $\text{hol}(\sigma^2) \mapsto U (\text{hol}(\sigma^2)) U^{-1}$. A direct computation shows:

$$[c_1](\sigma^2) \mapsto \frac{i}{2\pi} \log(\det (U (\text{hol}(\sigma^2)) U^{-1})) = \frac{i}{2\pi} \log(\det (\text{hol}(\sigma^2))) .$$

□

Proposition 7.2.3. $\langle c_1, \sigma^2 \rangle$ is independent of the basepoint used for the simplex.

Proposition 7.2.4. c_1 is closed.

Proof. It will be enough to compute $d^2 c_1$ for a given tetrahedron $\sigma^3 = [0123]$. By direct computation:

$$\begin{aligned}
\langle d c_1, [0123] \rangle &= \langle c_1, [123] \rangle - \langle c_1, [023] \rangle + \langle c_1, [013] \rangle - \langle c_1, [012] \rangle \\
&= \frac{i}{2\pi} \log \left\{ \det \{ \langle U, [12] \rangle \langle U, [23] \rangle \langle U, [31] \rangle \} \det \{ \langle U, [02] \rangle \langle U, [23] \rangle \langle U, [30] \rangle \}^{-1} \right. \\
&\quad \times \det \{ \langle U, [01] \rangle \langle U, [13] \rangle \langle U, [30] \rangle \} \det \{ \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \}^{-1} \Big\} \\
&= \frac{i}{2\pi} \log \left\{ \det \{ \langle U, [12] \rangle \langle U, [23] \rangle \langle U, [31] \rangle \} \det \{ \langle U, [03] \rangle \langle U, [32] \rangle \langle U, [32] \rangle \langle U, [20] \rangle \} \right. \\
&\quad \times \det \{ \langle U, [01] \rangle \langle U, [13] \rangle \langle U, [30] \rangle \} \det \{ \langle U, [02] \rangle \langle U, [21] \rangle \langle U, [10] \rangle \} \Big\} \\
&= 0,
\end{aligned}$$

where in the last step we used that the product of determinants is the determinant of the product. \square

Definition 7.2.5. Given a 2-dimensional simplicial complex K with fundamental class $[K] \in H_2(K)$ the **first Chern number** is $\langle c_1, [K] \rangle$, which can also be written as the sum: $\sum_{\sigma^2} \langle c_1, \sigma^2 \rangle$.

Proposition 7.2.6. Given $E \rightarrow K$ of rank k and $F \rightarrow K$ of rank l with first Chern classes $c_1(E)$ and $c_1(F)$ we have:

1. for the Whitney sum $E \oplus F$ the first Chern number is $c_1(E \oplus F) = c_1(E) + c_1(F)$, and
2. for the tensor product $E \otimes F$ the first Chern number is $c_1(E \otimes F) = l_1(E) + k c_1(F)$.

Proof. We will take the two parts of the proposition in turn:

1. It is enough to consider the Chern class on a single simplex $\sigma^2 = [012]$ with edge matrices labeled U for the bundle $E \rightarrow K$ and edge matrices labeled V for the bundle $F \rightarrow K$. A direct computation shows:

$$\begin{aligned}
\langle c_1(E \oplus F), \sigma^2 \rangle &= \frac{i}{2\pi} \log \{ \det \{ \langle U \oplus V, [01] \rangle \langle U \oplus V, [12] \rangle \langle U \oplus V, [20] \rangle \} \} \\
&= \frac{i}{2\pi} \log \{ \det \{ \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \oplus \langle V, [01] \rangle \langle V, [12] \rangle \langle V, [20] \rangle \} \} \\
&= \frac{i}{2\pi} \log \{ \det \{ \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \} \} \\
&\quad + \frac{i}{2\pi} \log \{ \det \{ \langle V, [01] \rangle \langle V, [12] \rangle \langle V, [20] \rangle \} \} \\
&= ([c_1(E)] + [c_1(F)]) (\sigma^2) .
\end{aligned}$$

2. Again, consider the same set-up as in part (1). We can again directly compute:

$$\langle c_1(E \otimes F), \sigma^2 \rangle = \frac{i}{2\pi} \log \{ \det \{ \langle U \otimes V, [01] \rangle \langle U \otimes V, [12] \rangle \langle U \otimes V, [20] \rangle \} \}$$

$$\begin{aligned}
&= \frac{i}{2\pi} \log \{ \det \{ \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \} \otimes \langle V, [01] \rangle \langle V, [12] \rangle \langle V, [20] \rangle \} \} \\
&= l \frac{i}{2\pi} \log \{ \det \{ \langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle \} \} \\
&\quad + k \frac{i}{2\pi} \log \{ \det \{ \langle V, [01] \rangle \langle V, [12] \rangle \langle V, [20] \rangle \} \} \\
&= l \langle c_1(E), [K] \rangle + k \langle c_1(F), [K] \rangle .
\end{aligned}$$

□

Proposition 7.2.7. *Let $L \rightarrow K$ be a $U(N)$ -bundle over a space without boundary, then $\langle c_1, [K] \rangle$ is an integer.*

Proof. For each edge, σ^1 , write the determinant of parallel transport matrix as $e^{i\alpha_{\sigma^1}}$ with $\alpha_{\sigma^1} \in \mathbb{R}$. Then the holonomy can be written as $e^{i \sum_{\sigma^1 \prec \sigma^2} \alpha_{\sigma^1}}$.

However, $|\sum_{\sigma^1 \prec \sigma^2} \alpha_{\sigma^1}|$ may not be between $-\pi$ and π . If it is not, choose an integer $N_{\sigma^2} \in \mathbb{Z}$ so that $|\sum_{\sigma^1 \prec \sigma^2} \alpha_{\sigma^1} + 2\pi N_{\sigma^2}| < \pi$. With this shift, we can give an explicit formula for c_1 :

$$\langle c_1, \sigma^2 \rangle = \frac{1}{2\pi} \sum_{\sigma^1 \prec \sigma^2} \alpha_{\sigma^1} + N_{\sigma^2} .$$

Now:

$$\begin{aligned}
\sum_{\sigma^2} \langle c_1, \sigma^2 \rangle &= \sum_{\sigma^2} \frac{1}{2\pi} \sum_{\sigma^1 \prec \sigma^2} \alpha_{\sigma^1} + N_{\sigma^2} \\
&= \sum_{\sigma^2} N_{\sigma^2} ,
\end{aligned}$$

since each α_{σ^1} appears in exactly two two-cells and has opposite sign in each. □

Corollary 7.2.8. *Let $L \rightarrow K$ be a $U(1)$ -bundle, if $|\log(\text{hol})| < \pi$ for all 2-simplices, then $\sum_{\sigma^2} c_1(\sigma^2) = 0$.*

Note that this corollary extends to bundles that are a Whitney sum of $U(1)$ -bundle.

7.3. Numerical Results

As a numerical test of our definition we computed the first Chern number of the tangent bundle on the unit sphere and torus with radii 1 and .25. We treated the surfaces as embedded in \mathbb{R}^3 , and computed the exact parallel transport, as if traveling on a geodesic, between the vertices of the mesh. For comparison we also computed the first Chern number using two different one simplex formulas and compiled the error with the true first Chern number in Table 7.1.

Our two comparison first Chern classes are defined in terms of our discrete curvature:

1. $\langle \tilde{c}_1(E), [012] \rangle = \frac{i}{2\pi} \text{tr}(\langle F, [012] \rangle \langle U, [20] \rangle) = \frac{i}{2\pi} \text{tr}(\langle \text{hol}, [012] \rangle - I)$ and

$$\begin{aligned}
2. \quad \langle c_1(E), [012] \rangle &= \frac{i}{4\pi} \text{tr}(\langle F, [012] \rangle \langle U, [20] \rangle - \langle F, [021] \rangle \langle U, [10] \rangle) \\
&= \frac{i}{4\pi} \text{tr}(\langle \text{hol}, [012] \rangle - \langle \text{hol}, [012] \rangle^\dagger)
\end{aligned}$$

While these formula are motivated by the smooth definition $c_1 = \text{tr } F$, we cannot use this construction for our discrete curvature since we would then not have gauge invariance. This is because our discrete curvature transforms as $\langle F, [012] \rangle \mapsto \langle h^{-1}, [0] \rangle \langle F, [012] \rangle \langle h, [2] \rangle$.

These two definitions do have share many of the axioms of the first Chern class. Namely, they obey the Whitney sum and tensor product formulas. However, the first definition does not even give a real-valued cochain. Indeed,

$$\overline{\langle \tilde{c}_1(E), [012] \rangle} = \frac{-i}{2\pi} \text{tr}(\langle \text{hol}^\dagger, [012] \rangle - \mathbb{1}) \neq -\frac{i}{2\pi} \text{tr}(\langle \text{hol}, [012] \rangle - \mathbb{1}) .$$

Since $\langle \text{hol}^\dagger, [012] \rangle \neq \langle \text{hol}, [012] \rangle$ we will generally not obtain a real-valued cochain. Numerical tests show that for the first Chern number on Riemann surfaces the imaginary error can be significant (see Table 7.1).

	N_2	Comparison 1		Comparison 2		Definition 7.2.1	
		real	imag	real	imag	real	imag
Sphere	328	0.0	3.35e-16	6.85e-4	4.23e-2	6.85e-4	0.0
	1218	0.0	1.22e-16	4.22e-5	1.09e-2	4.22e-5	0.0
	2624	0.0	9.61e-16	8.91e-6	5.04e-3	8.91e-6	0.0
	7922	0.0	1.53e-15	9.70e-7	7.67e-3	9.70e-7	0.0
Torus	2400	3.76e-17	1.75e-15	3.35e-12	-5.13e-2	3.35e-12	0.0

Table 7.1.: Numerical errors of the computed real and imaginary parts of the first Chern class definitions for the unit sphere and embedded torus. Comparison 1 is the definition: $\frac{i}{2\pi} \sum_\sigma \log(\det(\text{hol}(\sigma)))$, comparison 2 is the definition: $\frac{i}{2\pi} \sum_\sigma \text{tr}(\text{hol}(\sigma) - \mathbb{1})$, and Definition 7.2.1 is: $\frac{i}{4\pi} \sum_\sigma \text{tr}(\text{hol}(\sigma) - \text{hol}^{-1}(\sigma))$.

The errors in the real parts of the two alternative definitions can be understood in terms of expansion of the matrix exponential as in the discussion of curvature in the Preliminaries (Chapter 2). Since the holonomy around a region is equal to the exponential of the integral of curvature we have:

$$\text{hol} = e^{\int F} = \sum_{k=0}^{\infty} \frac{i^k (\int F)^k}{k!} .$$

Truncating the series at the first term yields:

$$\text{hol} = 1 + iF + O(F^2) ,$$

which after rearranging yields the first definition.

The second definition comes from the identity $\text{hol}^\dagger = e^{-i \int F}$. Then by subtracting holonomy

and its hermitian conjugate:

$$\begin{aligned}
\text{hol} - \text{hol}^\dagger &= \sum_{k=0}^{\infty} \frac{i^k F^k}{k!} - \sum_{k=0}^{\infty} \frac{(-i)^k F^k}{k!} \\
&= 2 \sum_{k=0}^{\infty} \frac{i^{2k+1} F^{2k+1}}{(2k+1)!} \\
&= 2iF + O(F^3) .
\end{aligned}$$

Since the error is expected to be $O(F^3)$ instead of $O(F^2)$ this explains the relative improvement of this definition. Also note that this definition is purely real.

7.4. Remaining Chern Classes

As noted in Chapter 2, the first Chern class determines all of the remaining Chern classes through the splitting principle. However, the splitting principle is mostly useful as a theoretical tool to prove theorems about characteristic classes. Instead, we constructed the following candidates for discrete Chern characters or Chern classes inspired by Chern-Weil theory:

$$\begin{aligned}
&\langle \text{ch}_2, [01234] \rangle \\
&= \sum_{\tau \in S^5} (-1)^{|\tau|} \text{tr} [\log(\langle U, [\tau(0)\tau(1)] \rangle \langle U, [\tau(1)\tau(2)] \rangle \langle U, [\tau(2)\tau(0)] \rangle) \\
&\quad \times \log(\langle U, [\tau(0)\tau(3)] \rangle \langle U, [\tau(3)\tau(4)] \rangle \langle U, [\tau(4)\tau(0)] \rangle)] , \tag{7.1}
\end{aligned}$$

$$\begin{aligned}
&\langle \text{ch}_2, [01234] \rangle \\
&= \sum_{\tau \in S^5} (-1)^{|\tau|} \text{tr} [(\log(\langle U, [\tau(0)\tau(1)] \rangle) + \log(\langle U, [\tau(1)\tau(2)] \rangle) + \log(\langle U, [\tau(2)\tau(0)] \rangle)) \\
&\quad \times (\log(\langle U, [\tau(0)\tau(3)] \rangle) + \log(\langle U, [\tau(3)\tau(4)] \rangle) + \log(\langle U, [\tau(4)\tau(0)] \rangle))] , \tag{7.2}
\end{aligned}$$

$$\langle \text{ch}_2, [01234] \rangle = \text{tr} [(\langle U, [01] \rangle \langle U, [12] \rangle \langle U, [20] \rangle - \mathbf{1}) (\langle U, [03] \rangle \langle U, [34] \rangle \langle U, [40] \rangle - \mathbf{1})] . \tag{7.3}$$

We then sought to check three properties:

1. That ch_2 is closed;
2. ch_2 obeyed the Whitney sum and tensor product rules: $\text{ch}_2(E \oplus F) = \text{ch}_2(E) + \text{ch}_2(F)$ and $\text{ch}_2(E \otimes F) = k\text{ch}_2(E) + l\text{ch}_2(F)$, where k is the rank of F and l is the rank of E ; and
3. ch_2 is gauge invariant.

We will also make use of the following lemma about traces of matrix logarithms:

Lemma 7.4.1. *Given non-singular matrices A and C which are $n \times n$ and B and D that are $m \times m$ then:*

1. $\text{tr} [\log(A \oplus B) \log(C \oplus D)] = \text{tr} [\log(A) \log(C)] + \text{tr} [\log(B) \log(D)]$
2. $\text{tr} [\log(A \otimes B) \log(C \otimes D)] = m \text{tr} [\log(A) \log(C)] + n \text{tr} [\log(B) \log(D)]$.

Proof. The fundamental piece of both of these theorems is the following facts about the logarithm of matrices: $\log(A \oplus B) = \log(A) \oplus \log(B)$ and $\log(A \otimes B) = \log(A) \otimes \log(B)$. Given that, the lemma follows due to trace properties. \square

Equation 7.1 For the proofs that follow it is helpful to rewrite this definition in the following form:

$$\langle \text{ch}_2, [01234] \rangle = \sum_{\tau \in S_5} (-1)^\tau \langle \tilde{\text{ch}}_2, [\tau(0)\tau(1)\tau(2)\tau(3)\tau(4)] \rangle ,$$

where $\tilde{\text{ch}}_2$ is defined as:

$$\langle \tilde{\text{ch}}_2, [01234] \rangle = \text{tr} [\log(\langle U, 01 \rangle \langle U, [12] \rangle \langle U, [20] \rangle) \log(\langle U, [03] \rangle \langle U, [34] \rangle \langle U, [40] \rangle)] .$$

This definition is manifestly gauge invariant and obeys the sum and tensor rules: $\text{ch}_2(E \oplus F) = \text{ch}_2(E) + \text{ch}_2(F)$ and $\text{ch}_2(E \otimes F) = k\text{ch}_2(E) + l\text{ch}_2(F)$ due to Lemma 7.4.1. It is, however is not closed and therefore cannot be an element of $H^4(K)$.

Equation 7.2 Like the previous trial definition, this trial definition obeys the sum and tensor product rules due to Lemma 7.4.1. However, this definition is not gauge invariant since for matrices A and B , $\log(AB) \neq \log(A) + \log(B)$. It is however, closed:

$$\begin{aligned} & \langle d^2 \text{ch}_2, [01234] \rangle \\ &= \sum_{\tau \in S^5} (-1)^{|\tau|} \text{tr} [(\log(\langle U, [\tau(0)\tau(1)] \rangle) + \log(\langle U, [\tau(1)\tau(2)] \rangle) + \log(\langle U, [\tau(2)\tau(0)] \rangle)) \\ & \quad \times (\log(\langle U, [\tau(0)\tau(3)] \rangle) + \log(\langle U, [\tau(3)\tau(4)] \rangle) + \log(\langle U, [\tau(4)\tau(0)] \rangle))] . \end{aligned}$$

In fact, it is exact. To see this it is helpful to note that each term in Equation 7.2 that has a repeated vertex (such as $\langle U, [01] \rangle \langle U, [02] \rangle$) appears an with a positive and negative sign exactly half of the time. This means that we can write Equation 7.2 as:

$$\begin{aligned} & \langle \text{ch}_2, [01234] \rangle \\ &= \text{tr} [\log(\langle U, [12] \rangle) \log(\langle U, [34] \rangle) + \log(\langle U, [23] \rangle) \log(\langle U, [41] \rangle) + \log(\langle U, [14] \rangle) \log(\langle U, [23] \rangle)] \\ & \quad - \text{tr} [\log(\langle U, [02] \rangle) \log(\langle U, [34] \rangle) + \log(\langle U, [01] \rangle) \log(\langle U, [40] \rangle) + \log(\langle U, [04] \rangle) \log(\langle U, [23] \rangle)] \\ & \quad + \text{tr} [\log(\langle U, [01] \rangle) \log(\langle U, [34] \rangle) + \log(\langle U, [13] \rangle) \log(\langle U, [40] \rangle) + \log(\langle U, [04] \rangle) \log(\langle U, [13] \rangle)] \end{aligned}$$

$$\begin{aligned}
& - \text{tr} [\log(\langle U, [01] \rangle) \log(\langle U, [24] \rangle) + \log(\langle U, [14] \rangle) \log(\langle U, [20] \rangle) + \log(\langle U, [04] \rangle) \log(\langle U, [12] \rangle)] \\
& + \text{tr} [\log(\langle U, [01] \rangle) \log(\langle U, [23] \rangle) + \log(\langle U, [12] \rangle) \log(\langle U, [30] \rangle) + \log(\langle U, [03] \rangle) \log(\langle U, [12] \rangle)] .
\end{aligned}$$

where we have re-grouped the to make apparent how this comes from an exterior derivative. The three-cochain whose exterior gives this is:

$$\text{tr} [\log(\langle U, [01] \rangle) \log(\langle U, [23] \rangle) + \log(\langle U, [12] \rangle) \log(\langle U, [30] \rangle) + \log(\langle U, [03] \rangle) \log(\langle U, [12] \rangle)] .$$

This means it cannot capture non-trivial second Chern number for boundary-less spaces. Importantly, to physical applications this definition would not be able to identify non-trivial second Chern number for $SU(2)$ gauge theories which is a important application of the second Chern class.

Equation 7.3 This definition is inspired by the holonomy definition of curvature much like the alternative definition of first Chern class we compared to our definition in the numerical tests and it has similar flaws. It is not closed and the sum over the entire space (which is related to the discrete version of the second Chern number) is commonly complex instead of purely real. While it is possible to ensure that this sum is purely real by taking the difference between the holonomy acquired by transversing around a simplex both clockwise and counter-clockwise (as we did in our numerical comparisons), the formula remains not closed and therefore not an element of cohomology. As the other definitions, this definition of the second Chern character obeys the Whitney sum and tensor product rules, though in a more simple manner than the previous definitions.

Chapter 8.

Reduction of Structure Group

Given a vector bundle $E \rightarrow K$ with connection with structure group G and a subgroup $H \subset G$, can all of the parallel transport matrices $\{U_{[v_j, v_i]}\}$ be chosen in H instead of in G ? More precisely, is there a gauge transformation $\{g_{v_i}\}$ that transforms each of the parallel transport matrices into elements of H ?

We will be especially interested in three special cases: (1) determining if H can be chosen to be the trivial group (thus showing the vector bundle is trivial), (2) finding the largest possible trivial subbundle of a given bundle, and (3) determining if a bundle can be decomposed in a Whitney sum of vector bundles.

If a positive solution exists to any of these problems, it means that the vector bundle can be subdivided into subbundles which immediately reduces the amount of work for many problems (which due to matrix multiplications scale with the rank of the vector bundle squared). Also, by separating the bundle into a direct sum decomposition that do not interact with each other, we can make problems on vector bundles immediately parallelizable.

8.1. Determining Triviality

We are interested in whether there is gauge transformation $\{g_{v_i}\}$ such that every parallel transport matrix $\{U_{[v_j, v_i]} = \mathbb{1}\}$.

Proposition 8.1.1. *Given a vector bundle $E \rightarrow K$ and a spanning tree, K_{tree} we can always trivialize all of the edges in the spanning tree.*

Proof. Begin at the root of the tree and apply the following in a breadth-first or depth-first manner:

1. Descend to a child vertex. The edge that you followed will have parallel transport matrix U . If the edge is oriented toward the child vertex apply the gauge transformation U at the child vertex, if the edge is oriented towards the parent vertex apply the gauge transformation U^{-1} .
2. Recurse to the children vertices following the breadth-first or depth-first search.

□

Note that treating the matrix operations as constant (since they do not vary once the structure group is determined) this algorithm have as a runtime that is proportional to the number of vertices. Depending on the desired output, real-world runtime can be improved by not trivializing each edge, but remembering what matrix operations would be required to trivialize the edge if one wanted to in the future.

Proposition 8.1.2. *$\text{hol}(\sigma^2) = \mathbb{1}$ for all $\sigma^2 \in K_2$ and $\text{hol}(\gamma) = \mathbb{1}$ for any representative $\gamma \in \pi_1(K)$, iff the bundle is trivial.*

Proof. Recall that holonomy transforms by the conjugation action under gauge transformations. If $E \rightarrow K$ is trivial this means every edge matrix can be chosen to be the identity matrix and so $\text{hol}(\sigma^1) = \text{hol}(\gamma) = \mathbb{1}$.

Suppose $\text{hol}(\sigma^1) = \text{hol}(\gamma) = \mathbb{1}$ for every σ^1 and representations of $\gamma \in \pi_1(K)$. Choose a spanning tree K_{tree} and trivialize according to Algorithm 8.1.1. Let σ^1 be an edge not in the spanning tree of K . Let γ be a loop containing σ^1 such that every other edge except σ^1 is in the spanning tree. By assumption $\mathbb{1} = \text{hol}(\gamma) = U_{\sigma^1}$. \square

An immediate corollary is possible for the case that the basespace is simply connected.

Corollary 8.1.3. *If K is simply connected and $\text{hol}(\sigma^2) = \mathbb{1}$ for every $\sigma^2 \in K_2$ then $E \rightarrow K$ is trivial.*

8.2. Determining Maximal Trivial Sub-bundle

More often, however, discrete vector bundles cannot be trivialized. In this case we are interested in the largest trivial sub bundle that can be found. We restrict our attention to bundles with structure group $U(n)$ (or $O(n)$ in the real case). The proof requires a lemma about linear algebra along with our algorithm for trivializing the vector bundle along a spanning tree.

Lemma 8.2.1. *Let $A \in U(n)$ (or $O(n)$) then if*

$$A [1, 0, 0, \dots, 0]^T = v, \quad (8.1)$$

for some $v \in \mathbb{C}^n$ with $\|v\|^2 = 1$, then there exists a Householder transformation H such that HA can be written as:

$$HA = \begin{pmatrix} 1 & [0, 0, \dots, 0] \\ [0, 0, \dots, 0]^T & B \end{pmatrix}$$

with $B \in U(n-1)$.

Proof. Since $A [1, 0, 0, \dots, 0]^T = v$ we know that A has the form:

$$A = \begin{pmatrix} v & C \end{pmatrix}$$

where $C \in \mathbb{C}^{n \times n-1}$.

Let H be the Householder transformation such that $H v = [1, 0, 0, \dots, 0]^T$. Applying this to Equation 8.1:

$$\left(\begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} HC \right) \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

Since C is unitary this means that:

$$\begin{pmatrix} (1, 0, \dots, 0) \\ C^\dagger H^\dagger \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

and therefore HC has the form:

$$HC = \begin{pmatrix} (0, 0, \dots, 0) \\ B \end{pmatrix}$$

□

Proposition 8.2.2. *Let $E \rightarrow K$ be a $U(n)$ or $O(n)$ bundle, then*

$$\dim(\ker(\nabla)) = \text{dimension of maximal trivial sub-bundle.}$$

Proof. Let s^1, s^2, \dots, s^k be the flat sections of $E \rightarrow K$. Choose a spanning tree K_{tree} . Apply a gauge transformation G at the root vertex, v_0 such that $s_{v_0}^1 = [1, 0, 0, \dots, 0]^T$, $s_{v_0}^2 = [0, 1, 0, \dots, 0]^T$, etc. Now apply the following algorithm recursively:

1. Beginning at a parent vertex v_i descend to a child vertex v_{i+1} . Without loss of generality assume that the edge is oriented towards the child vertex.
2. Since s^1 is a flat section we know that

$$U_{[i+1, i]} s_i^1 = s_{i+1}^1$$

By the recursion we have that $s_{[i]}^1 = [1, 0, 0, \dots, 0]^T$. Applying Lemma 8.2.1, we can apply a gauge transformation at vertex $[i+1]$ to transform $U_{[[i+1, i]}$ to

$$U_{[i+1, i]} = \begin{pmatrix} 1 & [0, 0, \dots, 0] \\ [0, 0, \dots, 0]^T & B \end{pmatrix}$$

3. Repeat recursively on all of the trivial sections for a given parent and on the submatrix B . Because of properties of Householder transformations, our subsequent gauge transformations will not disturb the general form of $U_{[i+1:i]}$.
4. Repeat recursively on the children vertices.

□

8.3. Determining Block Structure

Sometimes the discrete vector bundle is equivalent to a direct sum of vector bundles, in this case computational problems can be made more efficient by transforming the bundle into its direct sum decomposition. Again, the proof relies on a lemma about linear algebra and our algorithm to trivialize the bundle on a spanning tree.

Lemma 8.3.1. *Let A and B be unitary (orthogonal) matrices such that $[A, B] = 0$, then A and B can be put into the same block diagonal form.*

Proof. Since A and B commute they can be simultaneously diagonalized over the complex numbers. If A and B have no complex eigenvalues we are done. Otherwise assume that A has at least two complex eigenvalues λ and λ^* with eigenvectors V_λ and $V_\lambda^* = V_{\lambda^*}$. We know that these vectors are eigenvectors of B as well. Denote the eigenvalue of V_λ with respect to B as μ . We then find that for V_{λ^*} :

$$BV_{\lambda^*} = (BV_\lambda)^* = \mu^* V_\lambda^* = \mu^* V_{\lambda^*}$$

and so V_{λ^*} as eigenvalue μ^* with respect to B . Therefore with respect to the orthogonal basis $V_1 = \frac{V_\lambda + V_{\lambda^*}}{\sqrt{2}}$ and $V_2 = \frac{V_\lambda - V_{\lambda^*}}{i\sqrt{2}}$ we can write A and B in a block diagonal form with either 2x2 rotations matrices on the diagonal, or the 1x1 block with 1. □

Proposition 8.3.2. *Let $E \rightarrow K$ be a vector bundle structure group $O(n)$ or $U(n)$, the E can be decomposed into a direct sum of vector bundles if for any basis of loops, $\{\gamma_i\}$, $[hol(\gamma_j), hol(\gamma_k)] = 0$.*

Proof. If $[hol(\gamma_j), hol(\gamma_k)] = 0$, then for the structure group $U(n)$ the holonomies can be simultaneously diagonalized, for if the structure group is $O(n)$ then the holonomies can be put into canonical form. We then trivialize a spanning tree as laid out in Proposition 8.1.1. The remaining edge matrices must then have the same block structure as the holonomy of the loop that included it, and therefore there is a block decomposition for the vector bundle. □

Chapter 9.

Generalizing the Cheeger-Buser Inequalities to One-Laplacians on Hockey-Puck Domains

The Cheeger problem seeks to answer the question: “How does one partition a space?” One may hope that such a partition fairly divides the space into two regions. However, one may also hope to cut the space into two subspaces that are as disconnected from each other as possible that is, where the space is “thinnest.” Cheeger’s answer was to construct a constant that involves both of these conditions. However, determining the optimal cut is at least an exponentially difficult problem generally. Cheeger and Buser were able to show that this problem can be approximated by the lowest non-zero eigenvector and eigenvalue of the zero-cochain Laplacian.

We give an upper bound to the one-cochain Laplacian for a particular “hockey puck” shaped domain and show that this bound goes to zero as the region is “squeezed.” In addition, we share numerical experiments that exhibit the interplay of topology and geometry present in eigenvalues of the Laplacian.

9.1. Introduction

In Cheeger 1970 Cheeger established a lower bound for the lowest nonzero eigenvalue λ_1 of the Laplace-Beltrami operator on smooth manifolds. A decade later Buser Buser 1980 proved an upper bound for the same eigenvalue. These bounds were in terms of a geometric quantity h that Cheeger defined for manifolds and which is now commonly referred to as the *Cheeger constant* or *Cheeger number* of the manifold. Let M be a closed compact Riemannian manifold of dimension n . From [Cheeger 1970]:

$$h(M) := \inf_S \frac{|S|}{\min(|M_1|, |M_2|)}, \quad (9.1)$$

where the infimum is taken over all $(n - 1)$ -dimensional compact submanifolds $S \subset M$ dividing M into submanifolds M_1 and M_2 , with $M = M_1 \cup M_2$ and $\partial M_1 = \partial M_2 = S$. Here $|S|$ is the $(n - 1)$ -dimensional volume of S and $|M_i|$ is the n -dimensional volume of M_i .

In the concluding paragraph of Cheeger 1970 Cheeger wrote that “It would be of interest to generalize the argument given here to ... k -forms”. To our knowledge there are no known lower and upper bounds for the lowest nonzero eigenvalue of Laplacian on k -forms, Δ_k , for a general compact manifold. The only exception is the case of smooth surfaces, where these bounds follow trivially from the case of functions, as Cheeger pointed out in Cheeger 1970. For the k -forms case, bounds are known for some specific types of domains.

There is a discrete (combinatorial) version of the Cheeger constant for graphs Chung 1997. Let G be a graph with vertex set V . Following the notation in Chung 1997 define:

$$h(G) := \min_S \frac{|E(S, \bar{S})|}{\min(|S|, |\bar{S}|)}, \quad (9.2)$$

where the minimum is taken over all subsets $S \subset V$. Here $|E(S, \bar{S})|$ is the number of edges with one vertex in S and the other in its complement and $|S|$ is the number of vertices in S . (In another version of this definition, $|S|$ is the degree weighted number of vertices.) A naive algorithm to find the minimizing S is clearly exponential time since all possible vertex subsets are candidates. Alon 1986 showed that the eigenfunction corresponding to the first nonzero eigenvalue of the graph Laplacian provides an approximation to the desired vertex selection.

Note that in differential geometry Cheeger and Buser were interested in bounds on λ_1 . Cheeger defined the geometric quantity $h(M)$ for a manifold M and Cheeger and Buser gave bounds on λ_1 in terms of $h(M)$. In combinatorics the interest was in computing $h(G)$ as a characterization of the connectivity properties of graph G . In this case, λ_1 was used to give bounds on $h(G)$ and the corresponding eigenfunction was used to obtain an approximation to the optimal decomposition of the graph into S and its complement. Thus the aims and viewpoints in the manifold and graph case are in some sense dual of each other. In one case the eigenvalue is to be estimated and the estimation is in terms of a geometric quantity. In the other case the combinatorial quantity is to be estimated and the estimation is in terms of the eigenvalue.

9.2. Numerical Experiments

9.2.1. Hollow Hockey Puck

We created a hollow hockey puck, which was based on the cross-section Figure 9.5 but with corners removed. and numerically computed the spectrum of the 0-, 1-, and 2-Laplacians with natural boundary conditions. We then “squeezed” the center of the hockey puck to form a shape that was approaching a torus.

In Figure 9.1 are the graphs of the lowest eigenvalues of the Laplacians. Since there is no boundary, the non-zero eigenvalues of the 0- and 2-Laplacians are in bijective correspondence. That is:

$$\Delta_0 u = \lambda u \Leftrightarrow \Delta_2 * u = d d^* * u = * d * d u = * \Delta_0 u = \lambda * u$$

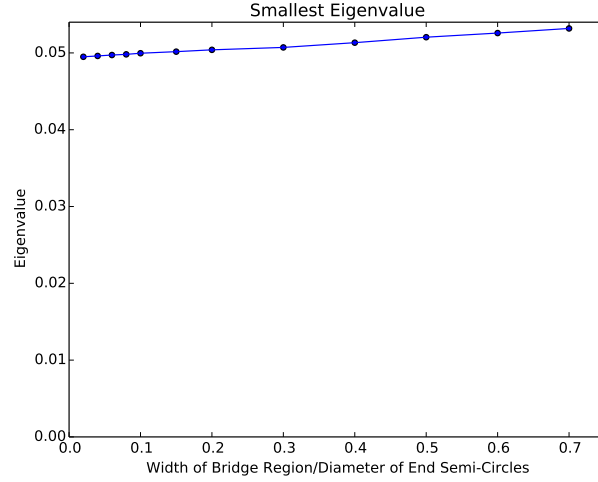


Figure 9.1.: First non-zero eigenvalue of the 1-Laplacian for the hollow hockey puck.

Furthermore, after the spectrum of the 1-Laplacian is the same as the 2-Laplacian with double the multiplicity. Since there are no new components forming, the first non-zero eigenvalue of the 0-Laplacian does not go to zero and hence the first non-zero eigenvalue of the 1-Laplacian does not approach 0.

9.2.2. Solid Hockey Puck

We created a solid hockey puck as with the same shape as the solid hockey puck and numerically computed the spectrum of the 0-, 1-, and 2-Laplacians with natural boundary conditions. We then “squeezed” the center of the hockey puck to form a shape that was approaching a solid torus. Figure 9.2, shows the eigenvector corresponding to the smallest non-zero eigenfunction is shown. This eigenfunction visually appears to be very similar to the harmonic eigenvector for this domain, which would also be “swirling around” much like this eigenvector.

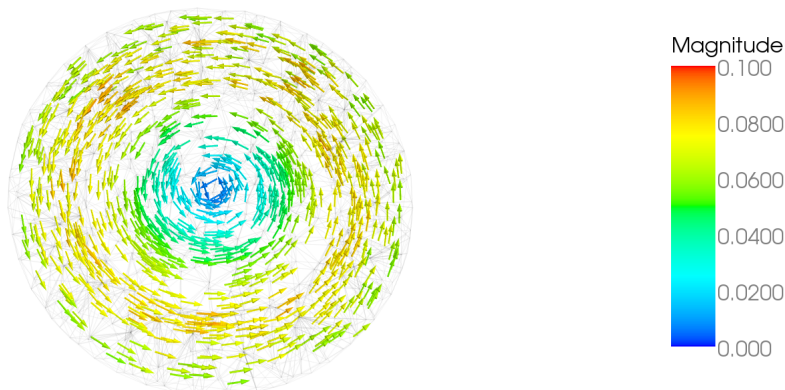


Figure 9.2.: Visualization of the “correct” eigenfunction (ω_0) when $\epsilon/2R = 0.2$.

This is in stark contrast to the next two eigenvectors which correspond to the (approximately equal) next larger eigenvalue (Figure 9.3). These eigenvectors are in the image of the exterior derivative, which can be seen from the visualizations. The vector field points from where the scalar field is larger to where it is smaller and levels off at the two ends. These eigenvectors are, in fact, rotations of each other.

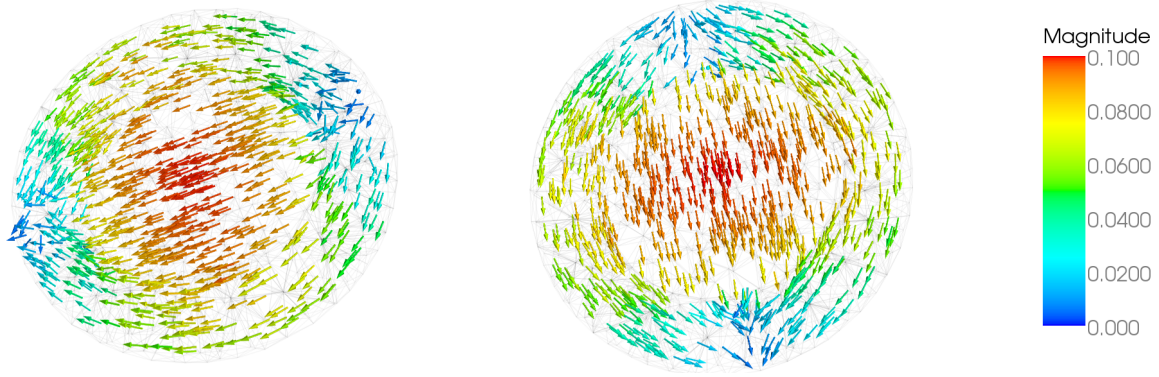


Figure 9.3.: Visualizations of the two “wrong” eigenvectors (ω_1, ω_2) when $\epsilon/2R = 0.2$.

We call these eigenvectors “wrong” because their eigenvalue is actually smaller than the nearly harmonic eigenvector’s eigenvalue when the domain is only slightly squeezed. In Figure 9.4a we plot the smallest eigenvalue of the domain and in Figure 9.4b we graph the eigenvalue for the “right” eigenvector as a function of space between the thinnest part of the “squeeze.” Since there are no new components forming, the second smallest eigenvalue of the 0-Laplacian does not get appreciably smaller as the hockey puck is squeezed. However, since a solid handle is being formed, there is an eigenvalue of the 1-Laplacian that does approach zero. Interestingly, this eigenvalue is not the smallest eigenvalue when the hockey puck is not very squeezed, but it eventually does become the smallest eigenvalue.

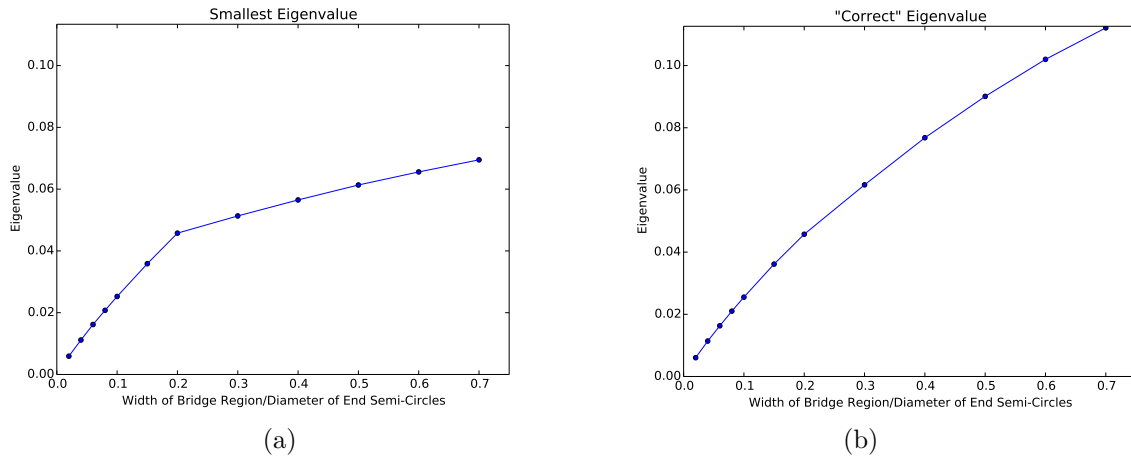


Figure 9.4.: (a) Smallest non-zero eigenvalue of the 1-Laplacian. (b) Eigenvalue of the “correct” eigenfunction ($\lambda \rightarrow 0$ as $\epsilon \rightarrow 0$).

Table 9.1 shows the value of these eigenvalues for a variety of different distances between the thinnest part of the domain as well as the value of the norm of the exterior derivative acting on both the “right” eigenvector (ω_0) and the “wrong” eigenvectors (ω_1 and ω_2). From this we can see that both ω_1 and ω_2 are closed and from the discussion in Section 2.1.4 must be in the image of the exterior derivative.

$\epsilon/2R$	λ_0	$\ d\omega_0\ $	λ_1	$\ d\omega_1\ $	λ_2	$\ d\omega_2\ $
0.70	0.069	0.000	0.070	0.000	0.113	0.336
0.60	0.066	0.000	0.066	0.000	0.102	0.320
0.5	0.061	0.000	0.061	0.000	0.091	0.301
0.40	0.056	0.000	0.057	0.000	0.077	0.278
0.30	0.051	0.000	0.051	0.000	0.062	0.249
0.20	0.046	0.214	0.046	0.000	0.046	0.000
0.15	0.036	0.189	0.043	0.000	0.043	0.000
0.10	0.025	0.159	0.039	0.000	0.040	0.000
0.06	0.016	0.127	0.036	0.000	0.037	0.000
0.02	0.006	0.077	0.033	0.000	0.034	0.000

Table 9.1.: Eigenvalue and the magnitude of the differential of the 3 eigenfunctions with smallest eigenvalues. The colored numbers track the “correct” eigenvalue.

9.3. An Upper Bound

We derive an upper bound for the lowest eigenvalue of the one-form Laplacian for “hockey puck” domains as a function of ϵ , the distance between the thinnest points on the hockey puck. The domain is made of a circular bulb and rectangular bridge as shown in Figure 9.5 which is then made into a solid of revolution.

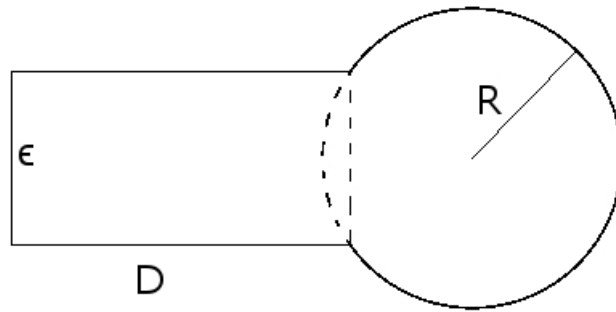


Figure 9.5.: Cheeger dumbbell, will be rotated about y axis

Proposition 9.3.1. *The smallest non-zero eigenvalue of the one Laplacian on the above domain*

is upper bounded by:

$$\lambda_0 \leq \frac{\frac{16L^6\pi}{15}\epsilon}{\frac{617\pi L^8}{3780}\epsilon + \frac{L^4 R^2}{6}\left(\frac{4R}{3\pi} + L\right)} = \frac{16/5L^2\pi\epsilon}{\text{Volume of ends} + (617/1260)\pi L^4\epsilon} \leq C\epsilon \quad \text{for small } \epsilon$$

Proof. Since the proof is quite long and technical, we will only sketch it here, for full details see Appendix A. Consider the one form:

$$\alpha = wr \, d\theta = \begin{cases} (r^3 - 8L/3r^2 + 2L^2r)r \, d\theta & r \leq L \\ D^4/3 \, d\theta & r > L \end{cases}. \quad (9.3)$$

This form obeys the natural boundary conditions for one forms as well as being second derivative continuous. We then compute the Rayleigh quotient of this test form:

$$\lambda \leq \frac{\omega^T \Delta_1 \omega}{\omega^T \omega},$$

which gives the result. □

As a further test, we computed the inner product between our test form and the numerically computed eigenvector corresponding to the smallest non-zero eigenvalue of the one-Laplacian for a variety of values of ϵ . As the domain becomes increasingly squeezed, the angle between our test eigenvector and the numerically computed eigenvector shrinks as shown in Figure 9.6.

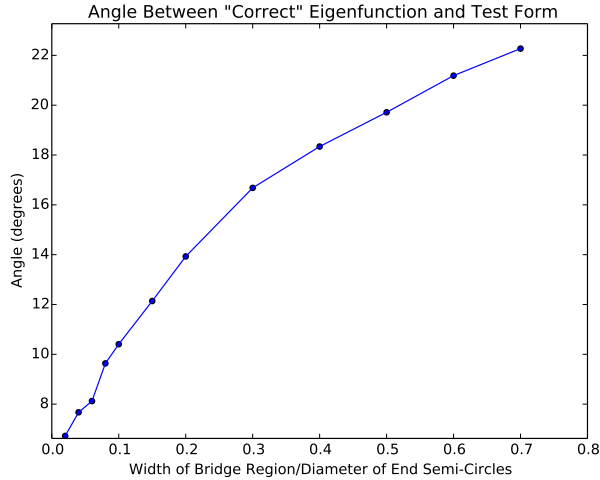


Figure 9.6.: Angle between our test eigenvector and the numerically computed smallest eigenvector corresponding to the lowest non-trivial eigenvalue (called the “correct eigenfunction”, see subsection on numerical results for hockey puck domain for an explanation of the naming) for the hockey-puck domain.

Chapter 10.

Conclusions and Future Aims

10.1. Conclusions

In this thesis we have presented a framework for discretizing field theories in a way that is independent of the underlying coordinates, developing a toolkit that can be used to model gauge theories. We began by improving the definition of the discrete dual exterior derivative, which allowed for the inclusion of more general boundary conditions into systems with quantities defined on the dual cells. This refining of the definition was also key to generalizing the proofs of adjointness of the exterior derivative and its differential to spaces with boundary.

Based on this definition we were able to derive discrete Euler-Lagrange equations for both full space-time discretizations as well as semi-discretizations and show that we are able to reproduce the equations of motion from the discrete action. Furthermore, we proved a limited version of Noether's theorem, though one that accounts for nearly all bosonic theories.

After defining discrete vector bundles in this context and providing some basic vector bundle operations such as the Whitney sum, tensor product and pullback we generalized the definition of connection to the discrete exterior derivative and showed that this definition reproduces the important properties that are found in the smooth setting. Combining the discrete covariant derivative with our work on discretizations of field theories produces Euler-Lagrange equations for systems with gauge fields.

We then showcased two applications in greater detail. The first was Yang-Mills and Yang-Mills coupled to a charged $U(n)$ -field. We demonstrated how to apply our discrete Euler-Lagrange equations to derive the equations of motion as well as derive the conserved charges. The second was two discretizations of Abelian Chern-Simons theory for boundary-less cellular complexes. In the fully-discrete model we were able to apply our machinery to demonstrate gauge invariance and charge conservation. The semi-discretization we verified the discrete theory shared the key properties as the smooth theory such as: gauge invariance and flux attachment as well as property of commutators that has implications to knot theory and topological properties of Abelian Chern-Simons.

We then turned our attention to topological invariants. We presented a definition of the first Chern class first presented in [Phillips 1985] for the case of $U(1)$ theories. We proved that this

definition is closed and obeys the relevant properties under tensor products, Whitney sums and pull-backs of discrete vector bundles. We demonstrated numerically that the first Chern class is related to the index of the connection Laplacian and finally provided a proof.

Our final chapters are problems in applied math. We first pose and provide solutions for a variety of reduction of structure group problems which seek to answer the question: “when can the structure group of a discrete vector bundle be reduced to a simpler one.” In Appendix A we provide an explicit example of each of our algorithms. In our chapter on generalizing the Cheeger-Buser inequalities we describe our numerical experiments on generalizing these inequalities beyond the scalar case.

10.2. Future Aims

Our work poses some new interesting questions. Particularly noteworthy are:

1. A discrete definition of the remaining Chern classes. This is probably the largest unanswered question discussed in our work, and while we have presented several possibilities they all have flaws that makes them presently incomplete solutions. Perhaps a relaxation of one of the properties we were seeking in the style of persistent homology will be able to provide a final answer to this.
2. The lack of definition of discrete spinors. Defining the spinor bundle is an important first step towards a discretization of fermions, which compose all of the matter in the universe.
3. Relatedly, without a definition of spinors or the remaining Chern classes, the proof of the general discrete index theorem remains open.
4. A proof of whether the Cheeger-Buser inequalities can be generalized and if they can what does is the generalization of the Cheeger number.
5. An integration of our techniques for discrete gauge theory with Regge calculus for curved space-times. This could have wide-ranging applications not just to computational general relativity but to material science systems where the medium is curved.

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Appendix A.

Proof of Upper Bound of One Form Laplacian Eigenvalue

Proposition A.0.1. *The smallest non-zero eigenvalue of the one Laplacian on the above domain is upper bounded by:*

$$\lambda_0 \leq \frac{\frac{16L^6\pi}{15}\epsilon}{\frac{617\pi L^8}{3780}\epsilon + \frac{L^4 R^2}{6}\left(\frac{4R}{3\pi} + L\right)} = \frac{16/5L^2\pi\epsilon}{\text{Volume of ends} + (617/1260)\pi L^4\epsilon} \leq C\epsilon \quad \text{for small } \epsilon$$

Proof. Consider the one form

$$\alpha = \omega r \, d\theta = \begin{cases} (r^3 - 8L/3r^2 + 2L^2r)r \, d\theta & r \leq L \\ D^4/3 \, d\theta & r > L \end{cases} \quad (\text{A.1})$$

First note that since $(L^3 - 8L^3/3 + 2L^3)L = L^4/3$ this form is continuous.

Also note that

$$\begin{aligned} d(\omega r \, d\theta) &= \frac{\partial(r\omega)}{\partial r} \, dr \wedge d\theta \\ &= \left(\omega + r \frac{\partial\omega}{\partial r}\right) dr \wedge d\theta \\ &= \left(\omega/r + \frac{\partial\omega}{\partial r}\right) dr \wedge r \, d\theta \\ &= 4(r - D)^2 \, dr \wedge r \, d\theta \end{aligned}$$

we see that this form is continuous under application of d

Also $\delta \omega r \, d\theta = * d \omega \, dz \wedge dr = 0$ it is continuous under δ .

Finally:

$$\begin{aligned} \Delta_1(\omega r \, d\theta) &= \delta d(\omega r \, d\theta) \\ &= *^{-1} d * \left(\omega/r + \frac{\partial\omega}{\partial r}\right) dr \wedge r \, d\theta \\ &= *^{-1} d\left(\omega/r + \frac{\partial\omega}{\partial r}\right) dz \end{aligned}$$

$$\begin{aligned}
&= *^{-1}\left(-\frac{\omega}{r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} + \frac{\partial^2\omega}{\partial r^2}\right)dr \wedge dz \\
&= -\left(\frac{\partial^2\omega}{\partial r^2} + \frac{1}{r}\frac{\partial\omega}{\partial r} - \frac{\omega}{r^2}\right)r d\theta \\
&= 8(D-r)r d\theta
\end{aligned}$$

so this form is continuous under application of the Laplacian.

We can further show that this form is C^2 by re-writing it in terms of the Cartesian coordinates (x, y, z) so

$$\alpha = \begin{cases} (x^2 + y^2 - 8L/3(x^2 + y^2)^{1/2} + 2L^2)(-y dy + y dx) & x^2 + y^2 \leq L^2 \\ D^4/3(x^2 + y^2)^{-1}(-y dy + y dx) & x^2 + y^2 > L^2 \end{cases} \quad (\text{A.2})$$

We now want to check that α is C^2 . We first check that the first derivative is continuous, which requires checking for continuity on the sphere $x^2 + y^2 = L^2$. In the region $x^2 + y^2 \leq L^2$:

$$\begin{aligned}
&\partial_x(-y[(x^2 + y^2)^{1/2} - 8L/3(x^2 + y^2) + 2L]) \\
&= -2xy + (8D/3)(xy)(x^2 + y^2)^{-1/2} \\
&= -2xy + (8/3)xy \quad \text{at } x^2 + y^2 = L^2 \\
&= 2/3xy .
\end{aligned}$$

And likewise for the partial derivative with respect to y :

$$\begin{aligned}
&\partial_y(-y[(x^2 + y^2)^{1/2} - 8L/3(x^2 + y^2) + 2L]) \\
&= -(x^2 + y^2) + 8L/3(x^2 + y^2)^{1/2} - 2L - y(2y - 8D/3y(x^2 + y^2)^{-1/2}) \\
&= -D^2 + 8L^2/3 - 2L - 2y^2 + 8/3y^2 \quad \text{at } x^2 + y^2 = L^2 \\
&= -L^2/3 + 2y^2/3 .
\end{aligned}$$

In the region $x^2 + y^2 \geq L^2$ the partial derivative with respect to x is:

$$\begin{aligned}
\partial_x(L^4/3(-y(x^2 + y^2)^{-1})) &= L^4/3 \frac{2xy}{(x^2 + y^2)^2} \\
&= 2/3xy \quad \text{at } x^2 + y^2 = L^2 ,
\end{aligned}$$

and y :

$$\begin{aligned}
\partial_y(L^4/3(-y(x^2 + y^2)^{-1})) &= -L^4/3 \frac{x^2 + y^2 + 2y^2}{(x^2 + y^2)^2} \\
&= 1/3(-D^2 + 2y^2/3) \quad \text{at } x^2 + y^2 = L^2 .
\end{aligned}$$

Now we will check that it is C^2 . We need to do all combinations of ∂_x and ∂_y . First is $\partial_x \partial_x$ in

the region $x^2 + y^2 \leq L^2$:

$$\begin{aligned}
& \partial_x^2(-y[(x^2 + y^2)^{1/2} - 8L/3(x^2 + y^2) + 2L]) \\
&= \partial_x(-2xy + (8L/3)(xy)(x^2 + y^2)^{-1/2}) \\
&= -2y + (8L/3)y(x^2 + y^2)^{1/2} - (8L/3)x^2y(x^2 + y^2)^{-3/2} \\
&= -2y + 8/3y - 8/3x^2y/L^2 \quad \text{at } x^2 + y^2 = L^2 \\
&= 2/3y - 8/3x^2y/L^2,
\end{aligned}$$

and in the region $x^2 + y^2 \geq L^2$:

$$\begin{aligned}
\partial_x^2(L^4/3(-y(x^2 + y^2)^{-1})) &= L^4/3\partial_x \frac{2xy}{(x^2 + y^2)^2} \\
&= (2L^4/3) \frac{y(x^2 + y^2)^2 - 4x^2y(x^2 + y^2)}{(x^2 + y^2)^4} \\
&= (2L^4/3)(y/L^4 - 4x^2y/L^6) \quad \text{at } x^2 + y^2 = L^2 \\
&= (2/3)y - 8x^2y/(3L^2).
\end{aligned}$$

Likewise for ∂_y . In the region $x^2 + y^2 \leq L^2$:

$$\begin{aligned}
& \partial_y^2(-y[(x^2 + y^2)^{1/2} - 8L/3(x^2 + y^2) + 2L]) \\
&= -\partial_y((x^2 + y^2) + 8L/3(x^2 + y^2)^{1/2} - 2L - y(2y - 8L/3y(x^2 + y^2)^{-1/2})) \\
&= -6y + 8L/3(y/(x^2 + y^2)^{1/2}) + 8L/3(2y(x^2 + y^2) - y^3/(x^2 + y^2)^{3/2}) \\
&= -6y + 8/3y + 8/3(2y - y^3/L^2) \quad \text{at } x^2 + y^2 = L^2 \\
&= 2y - 8/3(y^3/L^2),
\end{aligned}$$

and for the region $x^2 + y^2 \geq L^2$:

$$\begin{aligned}
\partial_y^2(L^4/3(-y(x^2 + y^2)^{-1})) &= -L^4/3\partial_y \frac{x^2 + y^2 + 2y^2}{(x^2 + y^2)^2} \\
&= -L^4/3 \frac{(x^2 + y^2)^2(2y) - (y^2 - x^2)4y(x^2 + y^2)}{(x^2 + y^2)^4}
\end{aligned}$$

Evaluating at $x^2 + y^2 = L^2$ gives:

$$\begin{aligned}
\partial_y^2(L^4/3(-y(x^2 + y^2)^{-1})) &= -1/3(2y - (4y^3 - (4y^3 - 4yL^2 + 4y^3)/L^2)) \\
&= -1/3(6y - 8y^3/L^2) = -2y + 8y^3/(3L^2).
\end{aligned}$$

Now we need to check the mixed partials. First $\partial_x\partial_y$. In the region $x^2 + y^2 \leq L^2$:

$$\partial_x\partial_y(-y[(x^2 + y^2)^{1/2} - (8L/3)(x^2 + y^2) + 2L])$$

$$\begin{aligned}
&= -\partial_x((x^2 + y^2) + 8L/3(x^2 + y^2)^{1/2}) - 2L - y(2y - (8L/3)y(x^2 + y^2)^{-1/2}) \\
&= -2x + (8L/3)x(x^2 + y^2)^{-1/2} - 8Ly^2x/3(x^2 + y^2)^{-3/2} \\
&= -2x + (8/3)x - (8/3)(xy^2)/L \quad \text{at } x^2 + y^2 = L^2 \\
&= (2/3)x - (8/3)(xy^2)/L,
\end{aligned}$$

and likewise in the region $x^2 + y^2 \geq L^2$:

$$\begin{aligned}
\partial_x \partial_y (L^4/3(-y(x^2 + y^2)^{-1})) &= -L^4/3 \partial_x \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \\
&= -(L^4/3) \frac{(x^2 + y^2)^2(2x) - (x^2 - y^2)(4x)(x^2 + y^2)}{(x^2 + y^2)^4} \\
&= -(L^4/3) \frac{-L^4 2x + 8xy^2 L^2}{L^8} \quad \text{at } x^2 + y^2 = L^2 \\
&= 2x/3 - (8/3)xy^2/L^2.
\end{aligned}$$

And finally for the mixed partial $\partial_y \partial_x$ in the region $x^2 + y^2 \leq L^2$:

$$\begin{aligned}
&\partial_y \partial_x (-y[(x^2 + y^2)^{1/2} - 8L/3(x^2 + y^2) + 2L]) \\
&= \partial_y (-2xy + (8D/3)(xy)(x^2 + y^2)^{-1/2}) \\
&= -2x + (8L/3)x(x^2 + y^2)^{-1/2} - xy^2(x^2 + y^2)^{-3/2} \\
&= -2x + (8/3)x - (8/3)xy^2/L^2 \quad \text{at } x^2 + y^2 = L^2 \\
&= (2/3)x - (8/3)xy^2/L^2,
\end{aligned}$$

and for the region $x^2 + y^2 \geq L^2$

$$\begin{aligned}
\partial_y \partial_x (L^4/3(-y(x^2 + y^2)^{-1})) &= L^4/3 \partial_y \frac{2xy}{(x^2 + y^2)^2} \\
&= L^4/3 \left(\frac{2x}{(x^2 + y^2)^2} - \frac{8xy^2}{(x^2 + y^2)^4} \right) \\
&= 2x/3 - (8/3)(xy^2/L^2).
\end{aligned}$$

Now we can apply our Rayleigh quotient:

$$\begin{aligned}
\langle wr \, d\theta, wr \, d\theta \rangle &= \int wr \, d\theta \wedge * wr \, d\theta \\
&= \int w^2 r \, d\theta \wedge dz \wedge dr \\
&= \int_{\text{bridge}} (r^3 - 8L/3r^2 + 2L^2r) r \, dr \, d\theta \, dz \\
&\quad + \frac{L^4}{3} \times \text{Volume of end rotated semi-circle} \\
&= \frac{617\pi L^8 \epsilon}{3780} + \frac{L^4}{3} \times \text{Volume of end rotated semi-circle}.
\end{aligned}$$

The volume of the end rotated semi-circle is given by:

$$V = \text{Area of object} \times \text{distance center of mass travels} = \frac{\pi R^2}{2} \left(\frac{4R}{3\pi} + L \right) .$$

$$\begin{aligned} \langle wr \, d\theta, \Delta_1 wr \, d\theta \rangle &= \int_{\text{bridge}} \Delta_1 wr \, d\theta \wedge * wr \, d\theta \\ &= \frac{16L^6 \pi \epsilon}{15} . \end{aligned}$$

So we arrive at the bound:

$$\lambda_0 \leq \frac{\frac{16L^6 \pi \epsilon}{15}}{\frac{617\pi L^8}{3780} \epsilon + \frac{L^4 R^2}{6} \left(\frac{4R}{3\pi} + L \right)} = \frac{16/5 L^2 \pi \epsilon}{\text{Volume of ends} + (617/1260) \pi L^4 \epsilon} \quad (\text{A.3})$$

$$\leq C\epsilon \quad \text{for small } \epsilon . \quad (\text{A.4})$$

□